Hořava Models as Palladium of Unitarity

霍拉瓦模型——幺正性的守护神

and Renormalizability in Quantum Gravity

与量子引力中的可重整性

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## 摘要

We give a review of UV renormalization of Hořava gravity (HG) models introduced as a remedy against violation of unitarity in quantum gravity theory. Projectable and non-projectable low-dimensional HG models and the spectra of their physical degrees of freedom are considered in the linearized approximation on flat spacetime background. The problem of regularity of their propagators depending on the choice of gauge-fixing procedure is discussed along with the role of this regularity in UV renormalization. With the choice of a special class of quasi-relativistic gauge conditions, perturbative renormalizability of projectable HG models is proven in any spacetime dimension and the status of renormalization in non-projectable models is briefly discussed. We show how the covariance of counterterms is provided within the class of background-covariant gauge conditions and show asymptotic freedom of $(2+1)$ -dimensional HG. We also present the calculation of beta-functions in $(3+1)$ -dimensional theory by the generalized Schwinger-DeWitt technique of universal functional traces and obtain fixed points of its renormalization group flow, some of them being good candidates for asymptotic freedom.

本文综述霍拉瓦引力 (HG) 模型的紫外重整化，这类模型是为解决量子引力理论中的幺正性破坏问题提出的。我们在平直时空背景的线性化近似下，讨论了可投影与不可投影低维 HG 模型及其物理自由度谱，探讨了传播子正则性依赖于规范固定选择的问题，以及该正则性在紫外重整化中的作用。通过选取一类特殊的准相对论规范条件，我们证明了可投影 HG 模型在任意时空维度下都是微扰可重整的，并简要讨论了不可投影模型的重整化现状。我们展示了在背景协变规范条件下如何保证 counterterms 的协变性，并证明了 $(2+1)$ 维 HG 具有渐近自由性质。我们还利用通用泛函迹的广义施温格-德维特技术计算了 $(3+1)$ 维理论的 $\beta$ 函数，得到了其重整化群流的不动点，其中部分不动点是渐近自由的良好候选。

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## Keywords

## 关键词

Hořava gravity - Renormalization group - Asymptotic freedom - Generalized Schwinger-DeWitt technique - Universal functional traces

霍拉瓦引力-重整化群-渐近自由-广义施温格-德维特技术-泛函迹

## Introduction

## 引言

The quest for renormalizable and perturbatively consistent in UV domain quantum gravity theory shows that such a theory can indeed be constructed by introducing higher-order curvature invariants [1-4], but it is doomed to violate unitarity in view of ghost modes associated with higher-order derivatives. In spite of various efforts to circumvent this problem or justify the presence of ghosts by special rules of handling them [5-8] or within the scope of string theory, nonlocal field models [9,10], etc., the most widespread point of view is that the absence of ghosts should be a criterion of selecting a healthy stable theory, and if this theory is also local, renormalizable, unitary, and consistent in UV limit, then there is a hope that it can also describe our nature.

> 寻找可重整、且在紫外区微扰一致的量子引力理论的研究表明，我们确实可以通过引入高阶曲率不变量构造出这样的理论 [1-4]，但由于高阶导数对应的鬼模态，该理论必然违背幺正性。尽管人们做了诸多尝试来绕过这一问题，或是通过特殊处理规则 [5-8]，或是在弦理论、非局域场模型 [9,10] 等框架内为鬼的存在辩护，但目前最普遍的观点认为: 无鬼应当是筛选健康稳定理论的判据，若一个理论同时满足局域性、可重整性、幺正性，且在紫外极限下自洽，那么它就有希望描述我们的真实自然。

Here we discuss the mechanism that can provide a combination of these properties, based on the breakthrough suggestion of [11, 12] that this can be achieved by dropping the requirement of Lorentz invariance and introducing in the field theory higher-order derivatives only with respect to spatial coordinates. This suggestion turned out to be very productive and used the notion, borrowed from condensed matter physics [13], of Lifshitz anisotropic scaling and scaling dimensions. Anisotropic scaling dimension replaces conventional physical dimensionality as a criterion of convergence of Feynman diagrams. Application of this criterion within simple power-counting arguments leads in [12] to an invention of the class of local, unitary quantum gravity models which are considered to be perturbatively renormalizable and, therefore, expected to be consistent in UV domain.

> 本文我们讨论能够结合上述所有性质的机制，该机制基于文献 [11,12] 的突破性思路: 放弃洛伦兹不变性的要求，在场论中仅对空间坐标引入高阶导数，即可实现这一目标。这一思路被证明极具产出性，它借用了凝聚态物理 [13] 里李夫希茨各向异性标度和标度维数的概念。各向异性标度维数取代了传统的物理维数，作为费曼图收敛性的判据。文献 [12] 通过简单幂次计数论证应用该判据，得到了一类局域幺正量子引力模型，这类模型被认为是微扰可重整的，因此有望在紫外区自洽。

The anticipation of such a fundamental breakthrough in high-energy domain of quantum gravity served as a motivation to explore low-energy consistency and phenomenology of Hořava's proposal; see [14, 15] for reviews. In particular, this has led to the identification of a Hořava gravity (HG) version - the so-called healthy non-projectable model [16] - which provides a consistent theory reproducing the phenomenology of general relativity (GR) at the distance scales where the latter has been tested. It was also realized that the theory never reduces to GR exactly: a certain amount of Lorentz invariance violation persists in the gravity sector at all distance scales [17]. This might have interesting implications for cosmological models of dark energy [18] and lead to astrophysical and cosmological constraints on the parameters of the theory [19-21]. To be phenomenologically viable, this model should include a mechanism ensuring Lorentz invariance in the sector of visible matter - the challenging issue addressed in [22-26].

> 量子引力高能领域取得这一基础性突破的预期，成为了人们探索霍拉瓦 proposal 低能自洽性与唯象学的动机；综述可参见 [14, 15]。相关研究尤其确定了霍拉瓦引力 (HG) 的一个版本——即所谓的健康非投射模型 [16]——该模型是一个自洽的理论，可以在广义相对论 (GR) 经实验检验的尺度上重现广义相对论的唯象结果。同时人们也认识到，该理论永远不会严格退化为广义相对论: 在所有尺度的引力部分都始终存在一定程度的洛伦兹不变性破缺 [17]。这可能给暗能量宇宙学模型带来有趣的启示 [18]，还可以给出天体物理和宇宙学对该理论参数的限制 [19-21]。为了满足唯象可行性要求，该模型需要引入一个机制来保证可见物质部分的洛伦兹不变性，这是文献 [22-26] 已经讨论过的挑战性问题。

Despite vast literature on HG there still remained a number of subtle issues in the fundamentals of Hořava proposal [12]. Namely, these are the problem of irregular propagators which might break the conventional BPHZ scheme of subtracting UV divergences by local counterterms and the problem of local gauge invariance of these counterterms, both of these issues especially inherent in HG construction. The point is that a general local gauge fixing in HG induces certain "irregular" contributions in the propagator of the metric that may spoil the convergence of the loop integrals [31]. As a consequence, a loop diagram that by a scaling argument should be finite can actually diverge and generate a counterterm not expected from the naive power-counting or, even, give rise to a nonlocal divergence. The key question was whether there exists a class of gauges where all propagators are regular.

> 尽管目前关于霍拉瓦引力的研究文献很多，但霍拉瓦 proposal[12] 的基础层面仍遗留了一系列微妙问题，具体包括: 可能破坏传统 BPHZ 方案用局域抵消项减除紫外发散的不规则传播子问题，以及这些抵消项的局域规范不变性问题，这两个问题在霍拉瓦引力的构造中尤为突出。问题的核心在于，霍拉瓦引力中一般的局域规范固定会给度规传播子带来某些“不规则”贡献，可能破坏圈积分的收敛性 [31]。结果就是，一个经标度论证本应有限的圈图实际上可能发散，产生朴素幂次计数无法预期的抵消项，甚至引发非局域发散。核心问题在于是否存在一类规范可以保证所有传播子都是规则的。

Another problem is the issue of covariant counterterms. It is well known that their manifest covariance as well as manifestly covariant nature of all intermediate calculations can be enforced by the use of the background field formalism in the class of background-covariant gauges. For various field models this was demonstrated within one-loop approximation [32-34], generically in perturbation theory [35,36] and in the framework of BRST cohomology methods with a special account of locality properties [37-39]. However, the extension of these results to local gauge theories with broken Lorentz invariance and effective field theories was not yet known. A clear demonstration that the BRST structure of renormalization in HG and other models with similar gauge invariance algebra is such that it reduces to renormalization of physical gauge fields separately from the renormalization of the BRST ghost sector was missing.

> 另一个问题是协变抵消项问题。众所周知，要让抵消项具有显协变性，且让所有中间计算都保持显协变，可以在背景协变规范类中使用背景场方法。对于各类场模型，这一点已经在一圈近似 [32-34]、一般微扰论框架 [35,36] 以及 BRST 上同调方法框架 (特殊考虑了局域性性质) 下得到了证明 [37-39]。但人们此前还不清楚能否将这些结果推广到洛伦兹不变性破缺的局域规范理论和有效场论中。此前也缺少明确的证明，来确认霍拉瓦引力以及其他具有类似规范不变性代数的模型中，重整化的 BRST 结构确实可以归结为物理规范场的重整化独立于 BRST 鬼区的重整化。

Both of the above problems have been successfully solved for the class of projectable Hořava models which have been proven to be perturbatively renormalizable in any spacetime dimension [40,41]. Moreover, in the series of papers [42-44] $(2+1)$-dimensional projectable HG was shown to be asymptotically free in UV limit, while its $(3+1)$-dimensional version turned out to have several interesting fixed points of renormalization group (RG) flow that can also be good candidates for asymptotic freedom. The goal of this chapter is to give a brief review of these results.

上述两类问题已针对可投影霍拉瓦模型类成功解决，这类模型已被证明在任意时空维度下都可微扰重整化 [40,41]。此外，在系列论文 [42-44] 中，(2+1) 维可投影霍拉瓦引力被证明在紫外极限下是渐近自由的，而其 (3+1) 维版本则具有多个值得关注的重整化群 (RG) 流不动点，这些不动点也可成为渐近自由的良好候选。本章的目的是对这些结果做简要综述。

Unfortunately, the projectable version of Hořava gravity does not reproduce GR at low energies (at least not within weak coupling) [17]. Nevertheless, it presents an interesting example of a theory sharing many properties of GR, such as a large gauge group of local spacetime transformations and the presence of gapless transverse-traceless excitations - gravitons - in dimensions $d = 3$ and higher. Working in the gauge with regular propagators we demonstrate that projectable Hořava gravity is perturbatively renormalizable in the strict sense.

遗憾的是，霍拉瓦引力的可投影版本无法在低能下重现广义相对论 (至少在弱耦合范围内不行)[17]。尽管如此，它仍是一个有意思的理论范例，该理论和广义相对论有许多共通性质，例如拥有大的局域时空变换规范群，且在 $d = 3$ 及更高维度存在无隙横迹激发——引力子。在正则传播子规范下我们证明，可投影霍拉瓦引力在严格意义上是可微扰重整化的。

For non-projectable HG model, which both theoretically and phenomenologically can be consistent with GR in the infrared domain [17], the regularity conditions of its propagators turn out to be violated. But these conditions are only sufficient rather than necessary for renormalizability, because a subtle mechanism of cancellation of harmful irregularities remains possible, as it was recently shown in [45]. So we also briefly discuss the source of this problem originating from the peculiarities of the canonical formalism of non-projectable HG.

对于不可投影霍拉瓦引力模型，其理论和唯象层面都可以在红外区和广义相对论自洽 [17]，该模型的传播子正则条件并不满足。但这些条件只是重整化可成立的充分条件而非必要条件，因为正如近期文献 [45] 所示，有害不规则性仍然可以通过微妙的抵消机制消除。因此我们也会简要讨论这一问题的来源，它根源于不可投影霍拉瓦引力正则形式的特殊性。

Renormalization of HG models undertaken in [42-44] raises another problem - enormous computational complexity associated with humongous amount of Feynman diagram vertices caused by the lack of Lorentz and full $(d+1)$ -dimensional diffeomorphism invariance of the theory. While in the renormalization of $(2+1)$ - dimensional HG, it was possible to use standard Feynman diagrams to reach the result [42]; in a similar $(3+1)$-dimensional case this becomes virtually impossible. It is enough to say that the inverse metric propagator in the background field formalism amounts to several hundred terms. So the method based on the combination of background field formalism and heat kernel technique [1, 46-50] becomes indispensable. This method provides the UV divergences not as expansion in powers of field perturbations, but as full nonlinear counterterms - local nonlinear functionals of the generic background field. Pioneering application of this method in Einstein theory [51] proved to be very efficient and now underlies the majority of results on renormalization of (super)gravitational models. The basic tool of this method is the heat equation kernel whose proper time expansion coefficients - the so-called HAMIDEW [52] or Gilkey-Seeley coefficients - carry a full information about UV divergences and can be systematically calculated.

文献 [42-44] 对霍拉瓦引力模型的重整化引出了另一个问题: 由于该理论不具备洛伦兹不变性和完整的 $(d+1)$ 维微分同胚不变性，费曼图顶点数量极多，带来了极高的计算复杂度。虽然对 $(2+1)$ 维霍拉瓦引力重整化而言，使用标准费曼图就能得到结果 [42]，但在类似的 (3+1) 维情形中这几乎不可能实现。只需说明一点: 背景场形式下的逆度量传播子就有数百项。因此，结合背景场形式与热核技术的方法 [1, 46-50] 成为了必不可少的工具。该方法给出的紫外发散不是场扰动的幂次展开，而是完整的非线性抵消项——即一般背景场的局域非线性泛函。该方法在爱因斯坦引力中的开创性应用 [51] 已被证明十分高效，如今是大多数 (超) 引力模型重整化结果的基础。该方法的基本工具是热方程核，其固有时展开系数——即所谓的 HAMIDEW 系数 [52](又称 Gilkey-Seeley 系数)——承载了紫外发散的全部信息，并且可以系统计算。

Despite powerful calculational advantages of the heat kernel method, its application to HG encounters the following major difficulty. It is directly applicable to the so-called minimal operators - second-order differential operators in which all spacetime derivatives are treated on equal footing and form covariant d'Alembertians. The existence of preferred time foliation in HG obviously violates this property. Several approaches have been put forward to circumvent this problem and extend the heat kernel method to Lifshitz-type theories [53-58]. Non-minimal operators - another difficulty in applications to HG models - arising in these models have higher-order derivative terms which are also not exhausted by powers of the spatial Laplacian $\Delta \equiv \gamma^{ij}\nabla_i\nabla_j$ . The principal symbol term of these operators is non-diagonal in derivatives whose indices are contracted with the tensor field indices. This difficulty was circumvented in [44] by the generalized Schwinger-DeWitt technique of the so-called universal functional traces (UFT) that was originally developed for spacetime covariant operators in [48, 49] (see also [59]). Here we will give a brief overview of this technique used for the calculation of the full set of beta-functions in $(3+1)$ - dimensional projectable HG.

尽管热核方法有强大的计算优势，将它应用于霍拉瓦引力仍存在一个主要困难。该方法仅直接适用于所谓的极小算子——即二阶微分算子，这类算子中所有时空导数地位平等，共同构成协达朗贝尔算子。霍拉瓦引力中存在优先时间叶状结构，显然违背了这一性质。目前已有多种方法被提出用来绕开这一问题，将热核方法推广到李夫希茨型理论 [53-58]。非极小算子是霍拉瓦引力模型应用中的另一难点，这类模型中出现的高阶导数项也不能完全由空间拉普拉斯算子 $\Delta \equiv \gamma^{ij}\nabla_i\nabla_j$ 的幂次表示。这类算子的主象征项在导数上是非对角的，导数的指标和张量场指标缩并。文献 [44] 通过广义施温格-德维特技巧即所谓的泛函迹 (UFT) 方法绕开了这一难点，该方法最初是为 [48, 49] 中的时空协变算子开发的 (也参见 [59])。本文将对这一技巧做简要综述，该技巧被用于计算 $(3+1)$ 维可投影霍拉瓦引力的完整 β 函数集。

The chapter is organized as follows. We begin with the Lifshitz idea of scaling which is anisotropic between space and time [13] and then apply it in section "Hořava Gravity Models" to quantum gravity as a remedy against violation of unitarity in the form of HG theory. After formulating the foliation-preserving diffeomorphism symmetry of HG models, we consider them in the linearized approximation on flat spacetime background along with their spectra of physical degrees of freedom in two low-dimensional cases. Then in section "Renormalizabil-ity and the Problem of Irregular Propagators" we discuss the problem of regularity of their propagators depending on the choice of gauge-fixing procedure and the role of this regularity in UV renormalization. With the choice of a special class of quasi-relativistic gauge conditions, we prove perturbative renormalizability of projectable HG models in any spacetime dimension and briefly dwell on the status of renormalization in non-projectable models. In section "BRST Structure of Renormalization and Covariance of Counterterms" we show how covariance of counterterms is provided within the class of background-covariant gauge conditions and then go over to the calculation of one-loop counterterms and renormalization group beta-functions in two low-dimensional theories. In this way in section ”Asymptotic Freedom in (2+1)-Dimensional Hořava Gravity” we show asymptotic freedom of $(2+1)$-dimensional HG and present in section ”One-Loop Beta-Functions of (3+1)-Dimensional Hořava Gravity” the calculation of beta-functions for $(3+1)$- dimensional theory by the generalized Schwinger-DeWitt technique of universal functional traces. After a brief discussion of fixed points and their properties, we finish the chapter with a concluding discussion.

本章结构安排如下。我们从空间与时间之间各向异性的李夫希兹标度思想 [13] 出发，随后在“霍拉瓦引力模型”一节中将其应用于量子引力，作为 HG 理论中解决幺正性破缺问题的方案。在确立 HG 模型的保叶状分形变不变性后，我们在平坦时空背景下对其做线性近似处理，并研究两类低维情况下其物理自由度的能谱。随后在“可重整性与不规则传播子问题”一节中，我们讨论了其传播子正则性的问题——该性质取决于规范固定过程的选择，以及该正则性在紫外重整化中的作用。选定一类特殊的准相对论规范条件后，我们证明了任意时空维度下可投影 HG 模型的微扰可重整性，并简要讨论了不可投影模型中的重整化状态。在“重整化的 BRST 结构与 counterterms 的协变性”一节中，我们展示了在背景协变规范条件下如何实现 counterterms 的协变性，随后进一步计算了两类低维理论中的单圈 counterterms 和重整化群 beta 函数。以此为基础，我们在“(2+1) 维霍拉瓦引力中的渐近自由”一节证明了 (2+1) 维 HG 的渐近自由性，并在“(3+1) 维霍拉瓦引力的单圈 beta 函数”一节中，利用通用泛函迹的广义施温格-德维特技术给出了 (3+1) 维理论 beta 函数的计算结果。在简要讨论不动点及其性质后，我们以总结性讨论结束本章。

## Lifshitz Theories with Anisotropic Scaling

## 各向异性标度的里夫希茨理论

As is well known, quantum gravity can be rendered UV renormalizable via introducing curvature-squared counterterms due to simple power-counting arguments of DeWitt [1, 32] later justified by a rigorous analysis of [2]. This however leads to higher-order spacetime derivatives in the Lagrangian and inevitably leads to ghost instabilities and loss of unitarity. The idea of salvation of unitarity in quantum gravity [11, 12] comes from Lifshitz work on phase transitions in condensed matter physics [13], suggesting the anisotropy between space and time. The ghost modes violating unitarity originate entirely from higher-order time derivatives. On the other hand, the UV convergence of Feynman diagrams can be improved by introducing higher-order derivatives only with respect to spatial coordinates, thus making the originally nonrenormalizable theory renormalizable without violation of unitarity.

众所周知，根据 DeWitt[1, 32] 的简单幂次计数论证 (后续经文献 [2] 严格分析验证)，量子引力可通过引入曲率平方 counterterm 实现 UV 可重整，这会在拉格朗日量中引入高阶时空导数，不可避免地会导致鬼场不稳定性并失去幺正性。量子引力中拯救幺正性的想法 [11, 12] 源自里夫希茨在凝聚态物理相变方面的工作 [13]，该工作提出了空间与时间之间的各向异性。破坏幺正性的鬼模完全来自高阶时间导数。另一方面，仅通过引入对空间坐标的高阶导数即可改善费曼图的 UV 收敛性，从而使原本不可重整的理论在不破坏幺正性的前提下变为可重整。

The implementation of this idea can be demonstrated on the example of Lifshitz scalar theory with the anisotropy between time and space. Consider the transition from usual relativistic invariant action to the action of a scalar field in $(d+1)$ spacetime dimensions

我们可以以存在时间与空间各向异性的里夫希茨标量理论为例展示这一思路的实现。考虑从常规相对论不变作用量切换到 $(d+1)$ 维时空中标量场的作用量

$$S=-\int \mathrm{d}^{d+1}x\partial_\mu\phi\partial^\mu\phi \Rightarrow S_L=\int \mathrm{d}t\ \mathrm{d}^d x\left(\dot{\phi}^2+\phi\mathbf{D}\phi\right), \tag{1}$$

where $\mathbf{D}$ is a higher-order differential operator in spatial derivatives of the form

其中 $\mathbf{D}$ 是空间导数的高阶微分算子，形式为

$$\mathbf{D}=-\frac{(-\Delta)^z}{(M^2)^{z-1}}+\dots,\ \Delta=\partial_i\partial^i,\ z>1. \tag{2}$$

Here the mass parameter $M^2$ is introduced to keep the physical dimensionality correct and lower derivative terms are denoted by dots.

此处引入质量参数 $M^2$ 是为了保证物理量纲正确，低阶导数项用省略号表示。

In contrast to Lorentz symmetry and usual scaling invariance, the new action becomes invariant under the symmetry broken down to $O(d)$ and a special anisotropic scaling,

与洛伦兹对称性和常规标度不变性不同，新作用量在破缺到 $O(d)$ 的对称性以及特殊各向异性标度下保持不变，

$$\begin{cases} x^\mu\mapsto b^{-1}x^\mu \\ \phi\mapsto b^{\frac{d-1}{2}}\phi,\ [\phi]=\frac{d-1}{2} \end{cases} \Rightarrow \begin{cases} t\mapsto b^{-z}t,\ x^i\mapsto b^{-1}x^i, \\ \phi\mapsto b^{\frac{d-z}{2}}\phi,\ [\phi]=\frac{d-z}{2}, \end{cases} \tag{3}$$

which allows one to introduce the notion of anisotropic scaling dimension of the field - a field $\phi$ with dimension $[\phi]=r$ transforms under the new scaling (3) as $\phi\mapsto b^r\phi$ . Accordingly we assign dimension -1 to the spatial coordinates $x^i$ and time has dimension $-z$ . Note that for $z\neq 1$ a scaling dimension is no longer equal to a physical dimension.

由此我们可以引入场的各向异性标度维数的概念: 维数为 $[\phi]=r$ 的场 $\phi$ 在新标度 (3) 下按 $\phi\mapsto b^r\phi$ 变换。相应地，我们将空间坐标 $x^i$ 的维数指定为-1，时间的维数为 $-z$ 。注意对于 $z\neq 1$ ，标度维数不再等于物理维数。

As will be clear in what follows, at the quantum level the renormalizability of such theories is determined by the same dimensional power-counting arguments as in Lorentz-invariant theories, but the role of dimensionality should be played by the scaling dimensionality rather than the physical one. This allows one to formulate a simple criterion for the choice of the parameter $z$ for which the theory becomes renormalizable.

下文将会说明，在量子层面，这类理论的可重整性由与洛伦兹不变理论相同的量纲幂次计数论证决定，但此处应当使用标度维数而非物理维数。据此我们可以得到参数 $z$ 的简单判据，满足该判据的理论可变为可重整理论。

It is natural that for nonlinear self-interacting fields the interaction terms contain highest spatial derivatives of the same order $2z$ , say in the cubic order in $\phi$ of a type $\sim\lambda\partial^{2z}\phi^3$ . Therefore, in view of the dimension of the integration measure $\left[\mathrm{d}t\ \mathrm{d}^d x\right] = -z - d$ and zero dimension of the interaction action, which we assume because the scaling symmetry is not supposed to be broken,

自然地，对于非线性自相互作用场，相互作用项包含阶数同为 $2z$ 的最高空间导数，例如 $\phi$ 三次项中类型为 $\sim \lambda\partial^{2z}\phi^3$ 的项。因此，考虑到积分测度 $\left[\mathrm{d}t\ \mathrm{d}^d x\right] = -z - d$ 的量纲，且我们假设标度对称性不发生破缺，因此相互作用作用量的量纲为零，

$$\left[\lambda\partial^{2z}\phi^3\right] = [\lambda] + 2z + \frac{3}{2}(d-z) = z + d, \tag{4}$$

the requirement of renormalizability - non-negative dimension of the coupling constant $\lambda$ - reads

可重整性要求——耦合常数 $\lambda$ 的量纲非负——可写为

$$[\lambda] = \frac{z-d}{2} \geq 0 \tag{5}$$

This leads to the critical value of $z$ at which the theory becomes renormalizable (and superrenormalizable beyond this value), $z_{\text{crit}} = d$ .

这给出了理论变为可重整的 $z$ 临界值 (超过该值后理论超可重整)，即 $z_{\text{crit}} = d$ 。

Let us now apply this idea in quantum gravity theory.

现在我们将这一思路应用到量子引力理论中。

## Hořava Gravity Models

## 霍拉瓦引力模型

The idea of Lifshitz anisotropic scaling implies an obvious mismatch between the number of spatial and temporal derivatives and leads to the loss of Lorentz invariance. In case of gravity this means that the theory cannot retain full local diffeomorphism invariance, and this local symmetry should be chosen to respect this higher-derivative structure in space vs. two derivatives in time. Such a symmetry can obviously be associated with the ADM split of the gravitational configuration space into spatial metric $\gamma_{ij}$ , lapse $N$ , and shift $N^i$ functions,

李夫希茨各向异性标度的观点表明空间导数与时间导数的数量存在明显不匹配，并会导致洛伦兹不变性破缺。就引力而言，这意味着该理论无法保留完整的局部微分同胚不变性，必须选择这种局部对称性来适配空间的高阶导数结构与时间的二阶导数结构。这种对称性显然可以和引力构型空间的 ADM 分解对应，分解得到空间度规 $\gamma_{ij}$ 、时移 $N$ 和位移 $N^i$ 函数，

$$ds^2 = -N^2\,\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\,\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\,\mathrm{d}t\right), i, j = 1, \ldots, d,$$

where $d$ is the dimensionality of space, which we consider rather general in order to learn how the properties of the model depend on spacetime dimensionality $D = d + 1$. Under this split the minimal truncation of the diffeomorphism invariance looks like the so-called foliation-preserving diffeomorphisms FDiffs under which spatial coordinates undergo generic time-dependent transformations accompanied by space-independent reparametrization of time,

其中 $d$ 是空间维数，为了研究模型性质如何依赖于时空维数 $D = d + 1$，我们这里考虑一般的空间维数。在该分解下，微分同胚不变性的最小截断就是所谓的保叶状结构微分同胚 FDiff，在这类变换下空间坐标会发生一般的含时变换，同时伴随时间的空间无关重参数化，

$$t \mapsto \tilde{t} = \tilde{t}(t), \; x^i \mapsto \tilde{x}^i = \tilde{x}^i(\mathbf{x}, t). \tag{6}$$

Lapse function, shift functions, spatial metric, and the extrinsic curvature

时移函数、位移函数、空间度规与外曲率

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \tag{7}$$

transform in such a way that the transformation laws for $\gamma_{ij}$ and $K_{ij}$ have a homogeneous tensor-type nature,

的变换方式满足 $\gamma_{ij}$ 和 $K_{ij}$ 的变换规律具有齐次张量性质，

$$N \mapsto \widetilde{N} = N\frac{dt}{d\tilde{t}}, \; N^i \mapsto \widetilde{N}^i = \left(N^j\frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{x}^i}{\partial t}\right)\frac{dt}{d\tilde{t}}, \tag{8}$$

$$\gamma_{ij} \mapsto \tilde{\gamma}_{ij} = \gamma_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}, \; K_{ij} \mapsto \widetilde{K}_{ij} = K_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}. \tag{9}$$

This easily enables to construct the invariant kinetic term of the action at most quadratic in time derivatives as a generic quadratic form in $K_{ij}$ and the rest of its Lagrangian can be constructed from local operators that transform as scalars under FDiffs and have dimension up to $2d$ - the maximal order of spatial derivatives,

由此我们可以很容易构造出作用量中对时间导数最高为二次的不变动能项，它是 $K_{ij}$ 上的一般二次型，拉格朗日的其余部分可以由满足 FDiff 下标量变换性质、维数不超过 $2d$ 的局域算符构造，其中 $2d$ 是空间导数的最高阶数，

$$S_{\rm HG}\left[\gamma_{ij}, N^j, N\right] = \frac{1}{2G}\int dt \, d^d x N\gamma^{1/2}\left(K_{ij}^2 - \lambda K^2 - \mathcal{V}\right). \tag{10}$$

Here $\lambda$ and $1/G = M^{d-1}$ are coupling constants (the latter for (3+1)-dimensional case is associated with the Planck mass squared $M_{\rm P}^2$), $K = \gamma^{ij}K_{ij}$, the dot stands for a time derivative, indices are raised and lowered by the spatial metric $\gamma_{ij}$, and the covariant spatial derivatives $\nabla_i$ are compatible with $\gamma_{ij}$. The potential term $\mathcal{V}$ consists of all allowed combinations of local invariants of scaling dimension up to $2d$ that are made of $\gamma_{ij}, N$, and their covariant derivatives $\nabla_i$. In this way one has a Lagrangian consisting of marginal and relevant operators with respect to the anisotropic scaling which in this sense is at least naively power-counting renormalizable if one prescribes the following scaling dimensions to the full set of field variables:

此处 $\lambda$ 和 $1/G = M^{d-1}$ 是耦合常数 ((3+1) 维情形下后者对应普朗克质量平方 $M_P^2$ )，$K = \gamma^{ij} K_{ij}$ ，点号代表时间导数，指标由空间度规 $\gamma_{ij}$ 升降，协变空间导数 $\nabla_i$ 与 $\gamma_{ij}$ 相容。势项 $\mathcal{V}$ 由所有满足要求的局域不变量组合构成，这些不变量的标度维数不超过 $2d$ ，由 $\gamma_{ij}, N$ 及其协变导数 $\nabla_i$ 构造得到。通过这种方式得到的拉格朗日，就各向异性标度而言仅包含边缘和相关算符；如果给所有场量指定如下标度维数，那么至少在 naive 幂次计数下该理论是可重整化的:

$$\left[\gamma_{ij}\right] = [N] = 0, \ \left[N^i\right] = d - 1, \ \left[K_{ij}\right] = d, \ [\mathcal{V}] = 2d. \tag{11}$$

With this choice, corresponding to the value $z = d$ of the parameter $z$ introduced above, the action has zero scaling dimension, $[S_{\mathrm{HG}}] = 0$ , in view of $\left[\mathrm{d}^d x\right] = [\mathrm{d}t] = -d$ . Note that both coupling constants $G$ and $\lambda$ also have zero scaling dimension in contrast to the situation with the physical dimension of $G, [G] = 0$ vs $[G]_{\text{phys}} = 1 - d$ .

在该选择下，对应上文引入的参数 $z$ 取值为 $z = d$ ，作用量的标度维数为零 $[S_{\mathrm{HG}}] = 0$ ，这可以由 $\left[\mathrm{d}^d x\right] = [\mathrm{d}t] = -d$ 看出。注意耦合常数 $G$ 和 $\lambda$ 的标度维数也都为零，这和 $G, [G] = 0$ 与 $[G]_{\text{phys}} = 1 - d$ 物理量纲的情况不同。

In the non-projectable Hořava gravity the lapse $N$ is postulated to be a function of both space and time. The status of renormalizability of this model is special, so that we postpone the discussion of this case until section ”Non-projectable Models.” So, to begin with, we focus on the projectable model where the lapse is a function of time only, $N = N(t)$ . Then the time reparameterization invariance allows one to set $N = 1$ leaving the time-dependent spatial diffeomorphisms as the remaining gauge transformations.

在非投影型霍拉瓦引力中，假设时移 $N$ 同时是空间和时间的函数。该模型的可重整化性质比较特殊，因此我们将这一情形的讨论推迟到“非投影型模型”一节。因此首先我们聚焦于投影型模型，其中时移仅为时间的函数，即 $N = N(t)$ 。此时时间重参数化不变性允许我们令 $N = 1$ ，仅保留含时空间微分同胚作为剩余规范变换。

## Projectable Models

## 可投影模型

For projectable models the potential term of the Hořava Lagrangian is a local function of spatial metric, curvature tensor, and its covariant derivatives. In the $(2 + 1)$ -dimensional case, $d = 2$ , the potential includes only two terms,

在可投影模型中，霍拉瓦拉格朗日量的势能项是空间度规、曲率张量及其协变导数的局部函数。在 $(2 + 1)$ 维情形下，$d = 2$ ，势能仅包含两项,

$$\mathcal{V}_{d=2} = 2\Lambda + \mu R^2 \tag{12}$$

because the linear in $R$ term is a total derivative in two dimensions and the Ricci tensor $R_{ij} = \frac{1}{2}\gamma_{ij} R$ reduces to the scalar curvature. Setting the cosmological constant $\Lambda$ to zero, we obtain a model with three marginal couplings $G, \lambda$ , and $\mu$ .

因为对于二维空间，$R$ 的线性项是全导数，且里奇张量 $R_{ij} = \frac{1}{2}\gamma_{ij}R$ 退化为标量曲率。将宇宙学常数 $\Lambda$ 设为零，我们得到一个含有三个边缘耦合 $G, \lambda$ 和 $\mu$ 的模型。

The second term of (12) together with the extrinsic-curvature terms is marginal under the scaling (3). They determine the UV behavior of the theory, in particular its renormalizability properties. The cosmological constant term with $\Lambda$ is a relevant deformation of the lowest dimension, which breaks anisotropic scaling in the infrared limit. We assume that it is tuned to zero in order to admit flat Minkowski spacetime as a solution.

式 (12) 的第二项与外曲率项在标度 (3) 下是边缘的。它们决定了该理论的紫外行为，尤其是它的可重整化性质。带有 $\Lambda$ 的宇宙学常数项是最低维的 relevant 形变，它在红外极限下破坏了各向异性标度。我们假设它被调节为零，以使平直闵可夫斯基时空成为理论的一个解。

To study the spectrum of linear perturbations around this background, we write

为了研究该背景下线性微扰的能谱，我们写出

$$\gamma_{ij} = \delta_{ij} + h_{ij} \tag{13}$$

and decompose the perturbations into scalar, vector, and transverse-traceless (TT) tensor parts,

并将微扰分解为标量、矢量和横无迹 (TT) 张量部分，

$$h_{ij} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right)\psi + \frac{\partial_i \partial_j}{\Delta} E + 2\partial_{(i} v_{j)} + t_{ij},\ N^i = \partial^i B + u^i, \tag{14}$$

$$\partial^i u_i = \partial^i v_i = 0,\ t^i_i = 0 = \partial^j t^i_j. \tag{15}$$

Expanding around flat spacetime and performing this decomposition, we obtain the quadratic action,

在平直时空附近展开并完成上述分解后，我们得到二次作用量，

$$S^{(2)}_{d=2} = \frac{1}{2G}\int \mathrm{d}t\, \mathrm{d}^2x \left[-\frac{1}{2}(\dot{v}_i - u_i)\Delta(\dot{v}_i - u_i) + \frac{\dot{\psi}^2}{4} + \frac{1}{4}\left(\dot{E} - 2\Delta B\right)^2 \right.$$
$$\left. -\frac{\lambda}{4}\left(\dot{\psi} + \dot{E} - 2\Delta B\right)^2 - \mu\psi\Delta^2\psi\right]. \tag{16}$$

Variation with respect to $u_i$ implies that there are no propagating modes in the vector sector. In the scalar sector we eliminate $E$ using the equation obtained upon variation with respect to $B$ and set as a gauge condition $B = 0$ afterwards. This yields

对 $u_i$ 变分表明，矢量部分不存在传播模式。在标量部分，我们利用对 $B$ 变分得到的方程消去 $E$，随后将 $B = 0$ 取为规范条件。由此得到

$$S^{(2)}_{d=2} = \frac{1}{2G}\int \mathrm{d}t\, \mathrm{d}^2x \left[\frac{1}{4}\frac{1-2\lambda}{1-\lambda}\dot{\psi}^2 - \mu\psi\Delta^2\psi\right], \tag{17}$$

so that unlike GR, which in $(2+1)$ dimensions does not possess any local degrees of freedom, Hořava gravity propagates a dynamical scalar mode. The latter has the dispersion relation,

因此，与广义相对论 (2+1 维广义相对论不具有任何局域自由度) 不同，霍拉瓦引力传播一个动力学标量模式。该模式满足色散关系:

$$\omega_s^2 = 4\mu \frac{1-\lambda}{1-2\lambda} k^4 \tag{18}$$

It is well-behaved (i.e., has positive kinetic term and is stable) if $G > 0, \mu > 0$ , and $\lambda < 1/2$ or $\lambda > 1$ .

当 $G > 0, \mu > 0$ ，且 $\lambda < 1/2$ 或 $\lambda > 1$ 时，该模式性质良好 (即动能项为正且稳定)。

In $d = 3$ , upon using the Bianchi identities, integrating by parts and noting that the Riemann tensor expresses in terms of the Ricci one, one finds the most general potential [28],

在 $d = 3$ 中，利用比安基恒等式、分部积分，再注意到黎曼张量可以用里奇张量表示，我们可以得到最一般的势能 [28]:

$$\mathcal{V}_{d=3} = 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij} R^{ij}$$

$$+ \upsilon_1 R^3 + \upsilon_2 R R_{ij} R^{ij} + \upsilon_3 R^i_j R^j_k R^k_i + \upsilon_4 \nabla_i R \nabla^i R + \upsilon_5 \nabla_i R_{jk} \nabla^i R^{jk}.$$

(19)

Here, $R_{ij}$ and $R$ are the Ricci tensor and Ricci scalar constructed from $\gamma_{ij}$ . In total, the theory contains 11 couplings: $G, \lambda, \Lambda, \eta, \mu_{1,2}$ , and $\upsilon_a, a = 1, \dots, 5$ . The terms in the second line of (19) together with the extrinsic-curvature terms in (10) are marginal under the scaling (3). They determine the UV behavior of the theory, in particular its renormalizability properties. The rest of the terms in (19) are relevant deformations. Among them the cosmological constant $\Lambda$ , which has the lowest dimension.

此处，$R_{ij}$ 和 $R$ 是由 $\gamma_{ij}$ 构造的里奇张量和里奇标量。该理论总共包含 11 个耦合常数: $G, \lambda, \Lambda, \eta, \mu_{1,2}$ 和 $\upsilon_a, a = 1, \dots, 5$ 。式 (19) 第二行的项与式 (10) 中的外曲率项在标度 (3) 下是边缘的，它们决定了理论的紫外行为，尤其是可重整化性质。式 (19) 中其余项都是 relevant 形变，其中宇宙学常数 $\Lambda$ 的维度最低。

Setting $\Lambda$ again to zero and expanding the action around flat Minkowski spacetime, we get its quadratic part

再次将 $\Lambda$ 设为零，并在平直闵可夫斯基时空附近展开作用量，我们得到它的二次部分

$$S^{(2)}_{d=3} = \frac{1}{2G} \int \mathrm{d}t \, \mathrm{d}^3 x \left\{ \left[ \frac{\dot{t}_{ij}^2}{4} + \frac{\eta}{4} t_{ij} \Delta t_{ij} - \frac{\mu_2}{4} t_{ij} \Delta^2 t_{ij} + \frac{\upsilon_5}{4} t_{ij} \Delta^3 t_{ij} \right] \right.$$

$$- \frac{1}{2} (\dot{\upsilon}_i - u_i) \Delta (\dot{\upsilon}_i - u_i)$$

$$+ \frac{\dot{\psi}^2}{2} + \frac{1}{4} (\dot{E} - 2\Delta B)^2 - \frac{\lambda}{4} (2\dot{\psi} + \dot{E} - 2\Delta B)^2$$

$$-\frac{\eta}{2}\psi\Delta\psi-\left(4\mu_1+\frac{3\mu_2}{2}\right)\psi\Delta^2\psi+\left(4v_4+\frac{3v_5}{2}\right)\psi\Delta^3\psi\Big\},$$

(20)

where the first line represents the tensor sector of transverse-traceless gravitons, the second sector corresponds to vector modes, and the last two lines form the scalar sector.

其中第一行对应横无迹引力子的张量部分，第二部分对应矢量模式，最后两行构成标量部分。

In order to identify the physical degrees of freedom, we perform the variation with respect to $u_i$ and $B$ and set them to zero afterwards by the gauge choice (three gauge conditions $u_i = B = 0$ for three diffeomorphisms). We obtain the equations,

为了确定物理自由度，我们对 $u_i$ 和 $B$ 做变分，随后通过规范选择 (对应三个微分同胚的三个规范条件 $u_i = B = 0$ ) 将它们设为零。我们得到方程:

$$\Delta\dot{v}_i=0,\ \Delta\left(\dot{E}-\frac{2\lambda}{1-\lambda}\dot{\psi}\right)=0. \tag{21}$$

The first one implies that the vector sector again does not contain any propagating modes. From the second equation in (21) we express $\dot{E}$ and substitute it back into (20) which yields the action for the propagating modes,

第一个方程表明，矢量部分依旧不包含任何传播模式。我们从式 (21) 的第二个方程解出 $\dot{E}$ ，将它代回式 (20)，得到传播模式的作用量:

$$S^{(2)}_{d=3}=\frac{M_P^2}{2}\int \mathrm{d}t\ \mathrm{d}^3x\Big\{\frac{\dot{t}_{ij}^2}{4}+\frac{t_{ij}}{4}\left[\eta\Delta-\mu_2\Delta^2+v_5\Delta^3\right]t_{ij}$$

$$+\frac{1-3\lambda}{1-\lambda}\frac{\dot{\psi}^2}{2}+\frac{1}{2}\psi\left[-\eta\Delta-(8\mu_1+3\mu_2)\,\Delta^2+(8v_4+3v_5)\,\Delta^3\right]\psi\Big\}.$$

(22)

In addition to the TT mode $t_{ij}$ , the theory propagates a "scalar graviton" $\psi$ . Both modes have positive-definite kinetic terms provided $G>0$ and $\lambda$ is either smaller than $1/3$ or larger than $1$ . The dispersion relations of two transverse-traceless modes and one scalar mode $\propto e^{-i\omega t+i\mathbf{kx}}$ are, respectively,

除 TT 模式 $t_{ij}$ 外，该理论还传播一个“标量引力子” $\psi$ 。只要 $G>0$ 满足且 $\lambda$ 小于 $1/3$ 或大于 $1$，两个模式的动能项都是正定的。两个横向无迹模式和一个标量模式 $\propto e^{-i\omega t+i\mathbf{kx}}$ 的色散关系分别为:

$$\omega_{tt}^2=\eta k^2+\mu_2k^4+v_5k^6, \tag{23}$$

$$\omega_s^2=\frac{1-\lambda}{1-3\lambda}\left(-\eta k^2+(8\mu_1+3\mu_2)\,k^4+(8v_4+3v_5)\,k^6\right). \tag{24}$$

This immediately raises a problem: the $k^2$ -term in the dispersion relation cannot be positive for both modes simultaneously. Thus, nonzero $\eta$ leads to an instability of the Minkowski background with respect to inhomogeneous perturbations. For positive values of the parameters $\mu_{1,2}$ and $v_{4,5}$ , the instability is cut off at large spatial momenta and therefore does not affect the UV properties of the theory. Moreover, we can stabilize the Minkowski spacetime by simply tuning $\eta$ to zero. However, in that case the dispersion relations of the TT mode and scalar gravitons are quadratic, $\omega \propto k^2$ , down to zero momentum, which prevents from recovering GR at low energies (One could try suppressing the instability with finite and positive $\eta$ by tuning $\lambda$ close to 1 . However, in this limit the theory becomes strongly coupled and the perturbative treatment breaks down [17, 30].).

这立刻引出一个问题: 色散关系中的 $k^2$ 项无法同时让两个模式为正。因此，非零的 $\eta$ 会导致闵氏背景对非均匀扰动产生不稳定性。对于参数 $\mu_{1,2}$ 和 $\upsilon_{4,5}$ 取正值的情况，不稳定性会在大空间动量处被截断，因此不会影响理论的紫外性质。此外，我们只需将 $\eta$ 调至零即可稳定闵氏时空。但在这种情况下，TT 模式和标量引力子的色散关系都是二次型 $\omega \propto k^2$ ，一直延伸到零动量，这导致理论无法在低能下还原广义相对论 (也可以尝试将 $\lambda$ 调至接近 1，通过取有限正的 $\eta$ 来抑制不稳定性，但在这个极限下理论会变成强耦合，微扰处理失效 [17, 30]。)。

We will be waving aside this difficulty of matching in the low energy domain the dispersion relations with those of general relativity [17], because we will restrict ourselves only with the analysis of renormalizability of the theory in high-energy domain. This analysis usually proceeds in the ”Euclidean” time obtained by the Wick rotation $t \mapsto \tau = it, N^j \mapsto N_{\mathrm{E}}^j = -iN^j$ , and the use of the relation $iS = -S_{\mathrm{E}}$ between the initial action and the Euclidean action $S_{\mathrm{E}}$ . The corresponding Euclidean action then differs from (10) only by the replacement of $t$ by $\tau$ and the sign of the potential term. At the quadratic level this amounts to flipping the signs of the terms containing $\mu_{1,2}, \upsilon_{4,5}$ in (20) and of the $\mu$ -term in (16).

我们这里撇开色散关系无法在低能区匹配广义相对论色散关系这个困难不谈 [17]，因为我们仅限定分析理论在高能区的可重整性。该分析通常在通过威克转动 $t \mapsto \tau = it, N^j \mapsto N_{\mathrm{E}}^j = -iN^j$ 得到的“欧氏时间”下进行，并且利用初始作用量和欧氏作用量之间的关系 $iS = -S_{\mathrm{E}}$ $S_{\mathrm{E}}$ 。对应的欧氏作用量与 (10) 式仅有的区别是将 $t$ 替换为 $\tau$ ，并改变势能项的符号。在二次项水平，这相当于翻转 (20) 式中含 $\mu_{1,2}, \upsilon_{4,5}$ 的项以及 (16) 式中 $\mu$ 项的符号。

## Renormalizability and the Problem of Irregular Propagators

## 可重整性与不规则传播子问题

The prospects of renormalization of the above models turn out, however, more complicated than it was originally anticipated in [12], because the proof of renor-malizability cannot really be accomplished by naive power-counting arguments. The point is that the degree of divergence of Feynman diagrams (denoted by $\mathcal{D}_{\text{div}}$ ) does not a priori provide correct renormalizability criteria of the BPHZ mechanism, because in the transition from Lorentz-invariant theories to Hořava models, it is now based on counting the anisotropic scaling dimension of their typical integrands,

然而，上述模型的重整化前景比 [12] 中最初预期的更为复杂，因为重整化的证明无法仅通过简单的幂次计数论证来完成。关键在于，费曼图的发散度 (用 $\mathcal{D}_{\text{div}}$ 表示) 并不能先验地为 BPHZ 机制提供正确的重整化准则，因为从洛伦兹不变理论过渡到霍拉瓦模型时，现在是基于计算其典型被积函数的各向异性标度维度

$$\mathcal{D}_{\text{div}}\left(\int\frac{d^{d+1}p}{\left(p^2\right)^N}\right)=1+d-2N\Rightarrow\mathcal{D}_{\text{div}}\left(\int\frac{d\omega d^d\mathbf{k}}{\left(A\omega^2+B\mathbf{k}^{2z}\right)^N}\right)=z+d-2zN,$$

(25)

with rather general coefficients $A$ and $B$ . Some of these coefficients might be zero and this creates a serious problem.

具有相当一般的系数 $A$ 和 $B$ 。其中一些系数可能为零，这会产生严重问题。

More generally this transition to the Lorentz non-invariant integrals over $(d+1)$ - dimensional loop momenta $p=(\omega,\mathbf{k})$ ,

更一般地，这种向关于 $(d+1)$ 维圈动量 $p=(\omega,\mathbf{k})$ 的非洛伦兹不变积分的转变，

$$\int\prod_{l=1}^{L}\mathrm{d}^{d+1}p^{(l)}\mathcal{F}_n(p)\prod_{m=1}^{M}\frac{1}{\left(P^{(m)}(p)\right)^2}$$
$$\Rightarrow\int\prod_{l=1}^{L}\mathrm{d}\omega^{(l)}\mathrm{d}^d\mathbf{k}^{(l)}\mathcal{F}_n(\omega,\mathbf{k})\prod_{m=1}^{M}\frac{1}{A_m\left(\Omega^{(m)}(\omega)\right)^2+B_m\left(\mathbf{K}^{(m)}(\mathbf{k})\right)^{2z}},$$

(26)results in propagators with various constant coefficients $A_m$ and $B_m$ (here $\Omega^{(m)}(\omega)$ and $\mathbf{K}^{(m)}(\mathbf{k})$ represent the momenta flowing in propagators as linear combinations of the full set of independent loop momenta $(\{\omega\},\{\mathbf{k}\})$) . It turns out that the rules of BPHZ subtraction of UV divergences are guaranteed only when $A_m$ and $B_m$ are both positive [40].

会得到带有不同常数系数 $A_m$ 和 $B_m$ 的传播子 (此处 $\Omega^{(m)}(\omega)$ 和 $\mathbf{K}^{(m)}(\mathbf{k})$ 表示传播子中流动的动量，为全部独立圈动量 $(\{\omega\},\{\mathbf{k}\})$) 的线性组合)。结果表明，只有当 $A_m$ 和 $B_m$ 均为正时，才能保证 BPHZ 减除紫外发散的规则成立 [40]。

To clarify the origin of this difficulty first note that a generic diagram contains subdivergences and thus can diverge despite $\mathcal{D}_{\text{div}}<0$ . Fortunately, as shown in [31], the combinatorics of the subtraction procedure in non-relativistic theories works essentially in the same way as in the relativistic case, and subdivergences are subtracted by counterterms introduced at the previous orders of the loop expansion. However, even in the absence of subdivergences, the convergence of a diagram with $\mathcal{D}_{\text{div}}<0$ is not trivial. Indeed, consider the $L$ -loop integral

为阐明这一困难的来源，首先注意到一般图包含次发散，因此即便 $\mathcal{D}_{\text{div}}<0$ 满足条件也可能发散。幸运的是，正如文献 [31] 所示，非相对论理论中减除过程的组合规则本质上与相对论情形相同，次发散可通过圈展开低阶引入的抵消项来减除。但即便不存在次发散，含 $\mathcal{D}_{\text{div}}<0$ 的图的收敛性也并非显然。例如，考虑这个 $L$ 圈积分

$$\int\mathrm{d}\omega\mathrm{d}^dk\int\left[\prod_{l=2}^{L}\mathrm{d}\omega^{(l)}\mathrm{d}^dk^{(l)}\right]f(\{\omega\},\{\mathbf{k}\})\equiv\int\mathrm{d}\omega\mathrm{d}^dk\widetilde{f}(\omega,k),\tag{27}$$

where we singled out from the full set $(\{\omega\},\{\mathbf{k}\})$ the first loop momentum $(\omega,\mathbf{k})$ and suppressed the dependence on external momenta. Assume for simplicity that $f$ is a scalar function (in general it can carry tensor indices corresponding to the external legs of the diagram). Because subdivergences are absent, the inner integral converges and gives a function $\widetilde{f}(\omega, k)$ which for $\mathbf{k} \mapsto b\mathbf{k}, \omega \mapsto b^2\omega$ scales as $b^{D_{\text{div}}-2d}$ . However, the latter can have the form

> 此处我们从完整集合 $(\{\omega\}, \{\mathbf{k}\})$ 中分离出了第一圈动量 $(\omega, \mathbf{k})$ ，并消去了对外部动量的依赖。为简化讨论，假设 $f$ 是标量函数 (一般情况下它可携带对应图外部腿的张量指标)。由于不存在次发散，内层积分收敛，得到函数 $\widetilde{f}(\omega, k)$ ，该函数在 $\mathbf{k} \mapsto b\mathbf{k}, \omega \mapsto b^2\omega$ 下的标度行为为 $b^{D_{\text{div}}-2d}$ 。但后者可写成如下形式

$$\widetilde{f}(\omega, k) \sim \omega^{-1\pm n} k^{\mathcal{D}_{\text{div}}-d\mp dn}, \; n > 0, \tag{28}$$

and the integral over frequency (momentum) will diverge, despite the fact that the $k$ - integral ( $\omega$ - integral) is finite. These are precisely the spurious divergences that arise if the propagators contain irregular contributions. Note that this problem is absent in Lorentz-invariant theories, where the function $\widetilde{f}$ can depend only on $\omega^2+k^2$ . In [40] it was proven that spurious divergences of the form (28) do not appear if all propagators in (26) have the regular form with all $A_m, B_m > 0$ . In that case $\mathcal{D}_{\text{div}} < 0$ indeed implies convergence of the diagram. The exact statement of [40] is as follows. Consider a diagram with $L$ loops and $\mathcal{D}_{\text{div}} < 0$ . Assume that all propagators in the diagram are regular in sense (26) and that if the momentum and frequency in any of the propagators are frozen, then the integral over remaining momenta and frequencies converges (i.e., subdivergences are absent). Then the whole diagram converges. This is the statement about convergence in the UV. Infrared divergences present a separate issue and must be regulated by an IR cutoff.

> 且对频率 (动量) 的积分会发散，尽管对 $k$ 的积分 ( $\omega$ 积分) 是有限的。这正是传播子包含不规则贡献时出现的伪发散。请注意，该问题在洛伦兹不变理论中不存在，在这类理论中函数 $\widetilde{f}$ 只能依赖于 $\omega^2 + k^2$ 。文献 [40] 已证明，若 (26) 中所有传播子都满足 $A_m, B_m > 0$ 、具备正则形式，那么 (28) 形式的伪发散就不会出现。此时 $\mathcal{D}_{\text{div}} < 0$ 确实保证了图收敛。[40] 的准确表述如下: 考虑一个含 $L$ 圈和 $\mathcal{D}_{\text{div}} < 0$ 的图，假设图中所有传播子都符合 (26) 定义的正则性，且冻结任意传播子中的动量和频率后，对剩余动量和频率的积分收敛 (即不存在子发散)，则整个图收敛。这是关于紫外收敛的结论。红外发散是另一类问题，必须通过红外截断来正则化。

The values of $A_m$ and $B_m$ , however, depend on the choice of the model and, moreover, on the choice of gauge conditions used in the Faddeev-Popov (or BRST) gauge-fixing procedure. This can be easily shown, say for the $d = 2$ case, by inverting the Hessian of the action (16) in the degenerate gauge $N^i = 0$ . The (ij, kl)-block of the full propagator in the momentum representation, $p = (\omega, \mathbf{k})$ , then contains the term

> 然而，$A_m$ 和 $B_m$ 的取值取决于模型选择，还取决于法捷耶夫-波波夫 (或 BRST) 规范固定过程中规范条件的选择。这一点很容易证明，例如在 $d = 2$ 情形下，对简并规范 $N^i = 0$ 中作用量 (16) 的黑塞矩阵求逆即可。动量空间中完全传播子的 (ij, kl) 块 $p = (\omega, \mathbf{k})$ 会包含如下项

$$\langle h_{ij}(p)\, h_{kl}(-p)\rangle = \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right)\frac{2G}{\omega^2} + \dots, \tag{29}$$

corresponding in the coordinate representation to the kernel $\langle h_{ij}(\tau, \mathbf{x})\, h_{kl}(\tau', \mathbf{x}')\rangle = -G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right)|\tau - \tau'$ … , which is singular for all $\tau - \tau'$ . This is in sharp contrast to local point-like singularities in Euclidean spacetime at $x = x'$ providing renormalization of UV divergences by local counterterms.

在坐标表示中对应核 $\langle h_{ij}(\tau,\mathbf{x})\, h_{kl}(\tau',\mathbf{x}')\rangle = -G\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-2\delta_{ij}\delta_{kl}\right)|\tau-\tau'|\,\delta^{(2)}(\mathbf{x}-\mathbf{x}')+\dots$，它对所有 $\tau-\tau'$ 都是奇异的。这与欧几里得时空下 $x=x'$ 处的局域点奇异形成鲜明对比，后者可通过局域抵消项完成紫外发散的重整化。

Another example is the nondegenerate gauge corresponding to the addition to the gauge-invariant Lagrangian of the gauge-breaking term

另一个例子是非简并规范，它对应在规范不变拉格朗日量中添加如下规范破缺项

$$\mathcal{L}_{\rm gf}=\frac{\sigma}{2G}F^i O_{ij} F^j, \tag{30}$$

where $F^i$ is a set of gauge condition functions linear in fields $N^i, h_{ij}$ , and their derivatives, while $\mathcal{O}_{ij}$ is an invertible gauge-fixing operator and $\sigma$ is a relevant gauge-fixing parameter. In order not to spoil the scaling properties of the action, this gauge-fixing term should have the total dimension of $2d$ , whereas all terms in $F^i$ and $\mathcal{O}_{ij}$ must scale appropriately. For the example of $d=2$ model with the nondegenerate gauge on shift vector variables, $F^i = N^i$ , the obvious choice is $\mathcal{O}_{ij} = -\delta_{ij}\Delta - \xi\partial_i\partial_j$ , and this leads to the propagator block $\langle N_i(p)N_j(-p)\rangle = G\left(\delta_{ij}-k_ik_j/k^2\right)/\sigma k^2+\dots$ corresponding again to nonlocal singularities $\sim \delta^{(1)}(\tau)\log|\mathbf{x}-\mathbf{x}'|$ , this time in space.

其中 $F^i$ 是一组对场 $N^i, h_{ij}$ 及其导数线性的规范条件函数，$\mathcal{O}_{ij}$ 是可逆规范固定算符，$\sigma$ 是相关的规范固定参数。为了不破坏作用量的标度性质，该规范固定项的总维度应为 $2d$，同时 $F^i$ 和 $\mathcal{O}_{ij}$ 中的所有项都必须具有合适的标度。对于位移矢量变量满足非简并规范的 $d=2$ 模型 $F^i=N^i$，显然的选择是 $\mathcal{O}_{ij}=-\delta_{ij}\Delta-\xi\partial_i\partial_j$，这会得到传播子块 $\langle N_i(p)N_j(-p)\rangle = G\left(\delta_{ij}-k_ik_j/k^2\right)/\sigma k^2+\dots$，其同样对应非局域奇点 $\sim \delta^{(1)}(\tau)\log|\mathbf{x}-\mathbf{x}'|$，只不过这次奇点出现在空间中。

These nonlocal singularities both in time and space are responsible for spurious divergences associated with irregular $1/\omega^2$ and $1/k^{2z}$ terms in the propagators and generically violate a conventional renormalization procedure, so that a subtle step in the proof of renormalizability consists in the search for a special class of gauges in which all propagators are regular.

这些同时存在于时间和空间中的非局域奇点会导致与不规则 $1/\omega^2$ 和 $1/k^{2z}$ 项相关的伪发散，通常会破坏常规重整化流程，因此证明可重整性的关键一步就是寻找一类特殊规范，使得所有传播子都是正则的。

This class of gauge conditions for generic spacetime dimensionality has been built in [40] as the following generalization of the relativistic gauge conditions which involve first-order time derivative of the shift functions, spatial derivatives of metric perturbations,

这类适用于任意时空维度的规范条件已在文献 [40] 中构造，它是对相对论规范条件的推广，包含位移函数的一阶时间导数和度规涨落的空间导数，

$$F^i=\dot{N}^i+\frac{1}{2\sigma}\left(O^{-1}\right)^{ij}\left(\partial^k h_{kj}-\lambda\partial_j h\right), \tag{31}$$

along with the generically nonlocal in space gauge-fixing operator in the gauge-breaking Lagrangian (30)

同时在规范破缺拉格朗日量 (30) 中引入了一般的空间非局域规范固定算符

$$O_{ij} = -(-\Delta)^{2-d}\left(\Delta\delta_{ij} + \xi\partial_i\partial_j\right)^{-1}. \tag{32}$$

With such a choice the elements of this two-parameter family of gauge conditions ( $\sigma$ and $\xi$ are free gauge-fixing parameters) have the following scaling dimensions:

采用该选择后，这个双参数规范条件族 ( $\sigma$ 和 $\xi$ 为自由规范固定参数) 的元素具有如下标度维度:

$$\left[F^i\right] = 2d - 1,\ \left[O_{ij}\right] = 2 - 2d, \tag{33}$$

which guarantee anisotropic scaling invariance of the full action. The identification of the parameter $\lambda$ with that of the kinetic term of the Hořava action (10) is not accidental - it allows, in particular, to avoid the cross $Nh$ -term in the quadratic part of the gauge-fixed action and thus simplifies the block structure of the full propagator of the theory. Moreover, this identification leaves a spatially nonlocal term $\sim \sigma\dot{N}^i O_{ij}\dot{N}^j/2G$ only in the quadratic part of the full gauge-fixed action and does not spoil the locality of vertices, which is necessary for correctness of BPHZ renormalization (see below).

这保证了完整作用量的各向异性标度不变性。将参数 $\lambda$ 和 Hořava 作用量 (10) 动能项的参数对应并非偶然——这种对应尤其可以避免规范固定作用量二次项中出现交叉 $Nh$ 项，从而简化理论完整传播子的分块结构。此外，这种对应仅会在完整规范固定作用量的二次项中保留一个空间非局域项 $\sim \sigma\dot{N}^i O_{ij}\dot{N}^j/2G$，且不会破坏顶点的局域性，而局域性是 BPHZ 重整化正确性的必要条件 (见下文)。

Let us show the regularity of this propagator in $d = 3$ case which we will consider in the high-energy limit, so that the coefficients of relevant deformation terms can be set to zero, $\eta = \mu_1 = \mu_2 = 0$ . For that one combines $\mathcal{L}_{gf}$ with the quadratic Lagrangian (20), $\eta = \mu_1 = \mu_2 = 0$ , and flips the sign of $v_{4,5}$ in consequence of the Wick rotation. Then, a straightforward calculation yields the nonzero components of propagators,

下面我们证明该传播子在 $d = 3$ 情况下的正则性，我们将在高能极限下讨论，因此可将相关形变项的系数设为零，即 $\eta = \mu_1 = \mu_2 = 0$ 。为此我们将 $\mathcal{L}_{gf}$ 与二次拉格朗日量 (20) 结合 $\eta = \mu_1 = \mu_2 = 0$ ，并根据 Wick 转动改变 $v_{4,5}$ 的符号，随后通过直接计算即可得到传播子的非零分量,

$$\left\langle N^i(p) N^j(-p)\right\rangle = \frac{Gk^2}{\sigma}\left(k^2\delta_{ij} - k_i k_j\right)\mathcal{P}_1(p) + \frac{\varkappa^2(1+\xi)k^2}{\sigma}k_i k_j \mathcal{P}_2(p), \tag{34}$$

$$\left\langle h_{ij}(p) h_{kl}(-p)\right\rangle = 2G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\mathcal{P}_{tt}(p)$$

$$-2G\delta_{ij}\delta_{kl}\left[\mathcal{P}_{tt}(p) - \frac{1-\lambda}{1-3\lambda}\mathcal{P}_s(p)\right]$$

$$-2\varkappa^2\left(\delta_{ik}\hat{k}_j\hat{k}_l + \delta_{il}\hat{k}_j\hat{k}_k + \delta_{jk}\hat{k}_i\hat{k}_l + \delta_{jl}\hat{k}_i\hat{k}_k\right)\left[\mathcal{P}_{tt}(p) - \mathcal{P}_1(p)\right]$$

$$+2\varkappa^2\left(\delta_{ij}\hat{k}_k\hat{k}_l + \hat{k}_i\hat{k}_j\delta_{kl}\right)\left[\mathcal{P}_{tt}(p) - \mathcal{P}_s(p)\right]$$

$$+2\varkappa^2\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l\left[\mathcal{P}_{tt}(p)+\frac{1-3\lambda}{1-\lambda}\mathcal{P}_s(p)-4\mathcal{P}_1(p)+\frac{2\mathcal{P}_2(p)}{1-\lambda}\right], \tag{35}$$

where $\hat{k}_i = k_i/\sqrt{k^2}$ is a spatial momentum normalized to unit vector and the pole structures are

其中 $\hat{k}_i = k_i/\sqrt{k^2}$ 是归一化为单位矢量的空间动量，极点结构为

$$\mathcal{P}_{tt} = \frac{1}{\omega^2+\upsilon_4 k^6},\ \mathcal{P}_s = \left[\omega^2+\frac{(8\upsilon_4+3\upsilon_5)(1-\lambda)}{1-3\lambda}k^6\right]^{-1}, \tag{36}$$

$$\mathcal{P}_1 = \left[\omega^2+\frac{k^6}{2\sigma}\right]^{-1},\ \mathcal{P}_2 = \left[\omega^2+\frac{(1-\lambda)(1+\xi)}{\sigma}k^6\right]^{-1}. \tag{37}$$

The first two structures correspond to the physical TT and scalar modes, cf. Eqs. (23)-(24), whereas the other two are gauge-dependent because they involve gauge-fixing parameters $\xi$ and $\sigma$ .

前两种极点结构对应物理 TT 模和标量模，参见式 (23)-(24)，另外两种则是规范依赖的，因为它们包含规范固定参数 $\xi$ 和 $\sigma$ 。

The propagator (34) is obviously regular. For the terms in the last three lines of (35), the situation is subtler. One may worry that the unit vectors entering them contain factors $k$ in the denominator and apparently violate the regularity condition $A_m \neq 0$ in (26). However, we observe that the combinations in the square brackets in these terms vanish at $k = 0, \omega \neq 0$ . Besides, they depend on the spatial momentum through $k^6$ . This implies that when the worrisome terms are written as ratios of polynomials, their numerators are at least proportional to $k^6$ , which cancels all powers of $k$ in the denominator. This cancellation is in fact guaranteed by the regularity of the propagator $\langle h_{ij}h_{kl}\rangle$ at $k \to 0, \omega$ -fixed; this, in turn, follows from the regular structure of the kinetic term of $\mathcal{L}+\mathcal{L}_{\rm gf}$ for $h_{ij}$ in this limit.

传播子 (34) 显然是正则的。对于 (35) 最后三行中的项，情况则更为微妙。人们可能会担心，其中出现的单位向量分母含有因子 $k$ ，这显然违反了 (26) 中的正则性条件 $A_m \neq 0$ 。但我们发现，这些项中方括号内的组合在 $k = 0, \omega \neq 0$ 处等于零。此外，它们对空间动量的依赖通过 $k^6$ 体现。这意味着，当这些令人担忧的项写为多项式比值时，它们的分子至少与 $k^6$ 成正比，这抵消了分母中所有的 $k$ 幂次。这种抵消实际上由传播子 $\langle h_{ij}h_{kl}\rangle$ 在 $k \to 0, \omega$ 固定时的正则性保证；而这又反过来源于该极限下 $\mathcal{L}+\mathcal{L}_{\rm gf}$ 对应 $h_{ij}$ 的动能项的正则结构。

The ghost sector of the theory can be built by standard rules of the Faddeev-Popov or BRST gauge-fixing procedure. The ghost action is the bilinear combination of the anticommuting ghost fields $c^i$ and $\bar{c}_j$ ,

该理论的鬼场部分可以按照 Faddeev-Popov 或 BRST 规范固定程序的标准规则构建。鬼场作用量是反对易鬼场 $c^i$ 与 $\bar{c}_j$ 的双线性组合，

$$S_{gh} = -\frac{1}{G}\int d\tau d^2x \bar{c}_i\left(\delta^c F^i\right), \tag{38}$$

where $\delta^c F^i$ is the linear transformation of gauge-fixing functions - their linearized foliation-preserving diffeomorphism $\delta^f F^i$ with the vector parameter $f^i$ identified with the ghost $c^i$ . With the finite diffeomorphism given by Eqs. (6),(8), and (9) its linearized version, $\tilde{x}^i = x^i + f^i(x,\tau)$ , reads for $h_{ij}$ as the Lie derivative with respect to $f^i, \delta^f h_{ij} = \mathcal{L}_f\gamma_{ij}$ , whereas for $N^i$ its $\mathcal{L}_f N^i$ is also amended by $\dot{f}^i$ ,

其中 $\delta^c F^i$ 是规范固定函数的线性变换——即保持叶状结构的微分同胚 $\delta^f F^i$ 的线性化形式，矢量参数 $f^i$ 对应鬼场 $c^i$ 。在由式 (6)、(8) 和 (9) 给出的有限微分同胚下，其线性化版本 $\tilde{x}^i = x^i + f^i(x,\tau)$ 对 $h_{ij}$ 来说就是关于 $f^i, \delta^f h_{ij} = \mathcal{L}_f \gamma_{ij}$ 的李导数，而对 $N^i$ 来说，它的 $\mathcal{L}_f N^i$ 还会被 $\dot{f}^i$ 修正，

$$\delta^f h_{ij} = \partial_i f^k (\delta_{jk} + h_{jk}) + \partial_j f^k (\delta_{ik} + h_{ik}) + f^k \partial_k h_{ij}, \tag{39}$$

$$\delta^f N^i = \dot{f}^i + f^j \partial_j N^i - N^j \partial_j f^i. \tag{40}$$

Thus, for $d = 3$ in the gauge (31)-(32) the ghost action reads

因此，对于规范 (31)-(32) 下的 $d = 3$ ，鬼场作用量可写为

$$S_{gh} = \frac{1}{G} \int \mathrm{d}\tau \mathrm{d}^2 x \Big[ \dot{\bar{c}}_i \dot{c}^i - \frac{1}{2\sigma} \bar{c}_i \Delta^3 c^i + \frac{1 - 2\lambda + 2\xi(1-\lambda)}{2\sigma} \partial_i \bar{c}_i \Delta^2 \partial_j c^j$$

$$+ \Delta \dot{\bar{c}}_i \partial_j c^i N^j - \Delta \dot{\bar{c}}_i c^j \partial_j N^i + \frac{1}{\sigma} \Delta^2 \partial_j \bar{c}_i \left( \partial_{(i} c^k h_{j)k} + \frac{1}{2} c^k \partial_k h_{ij} \right)$$

$$- \frac{\lambda(1+\xi)}{\sigma} \Delta^2 \partial_i \bar{c}_i \left( \partial_k c^l h_{lk} + \frac{1}{2} c^l \partial_l h \right)$$

$$+ \frac{\xi}{\sigma} \Delta \partial_i \partial_j \partial_k \bar{c}_i \left( \partial_j c^l h_{kl} + \frac{1}{2} c^l \partial_l h_{jk} \right) \Big].$$

(41)

This action is invariant under the anisotropic scaling (3) with the assignment of zero scaling dimension to the ghosts,

该作用量在标度变换 (3) 的各向异性标度下不变，其中鬼场的标度维数被赋值为零，

$$[c^i] = [\bar{c}_i] = 0 \tag{42}$$

and it gives rise to the ghost propagator also satisfying the regularity condition,

它给出的鬼传播子也同样满足正则性条件，

$$\langle c^i(p) \bar{c}_j(-p) \rangle = \varkappa^2 \delta_{ij} \mathcal{P}_1(p) + \varkappa^2 \hat{k}_i \hat{k}_j [\mathcal{P}_2(p) - \mathcal{P}_1(p)]. \tag{43}$$

Similar properties of Hořava gravity models in this class of gauges hold in all higher dimensions, and we are now ready to prove their UV renormalizability.

这类规范下的霍拉瓦引力模型在所有更高维度中都具备类似性质，现在我们已经可以证明它们的紫外可重整性了。

## Proof of Renormalizability

## 可重整性证明

For the full set of all quantum fields $\phi = h_{ij}, N^i, c^i, \bar{c}_i$ every Feynman graph is characterized by the following set of parameters:

对于全部量子场集合 $\phi = h_{ij}, N^i, c^i, \bar{c}_i$ ，每个费曼图都可以用以下参数集合表征:

$P_{hh}$ – number of $\langle h_{ij} h_{kl} \rangle$ propagators

$P_{hh}$ – $\langle h_{ij} h_{kl} \rangle$ 传播子数量

$P_{NN}$ - number of $\langle N^i N^j \rangle$ propagators

$P_{NN}$ - $\langle N^i N^j \rangle$ 传播子数量

$P_{c\bar{c}}$ – number of the ghost propagators

$P_{c\bar{c}}$ – 鬼场传播子数量

$V_{[h]}$ - number of vertices involving only the $h_{ij}$ -fields

$V_{[h]}$ - 仅含 $h_{ij}$ 场的顶点数量

- number of vertices with an arbitrary number of $h$ -legs and a single $N$ -leg

- 含任意数量 $h$ 支脚、单 $N$ 支脚的顶点数量

$V_{[h]NN}$ - number of vertices with an arbitrary number of $h$ -legs and two $N$ -legs

$V_{[h]NN}$ - 含任意数量 $h$ 支脚、两个 $N$ 支脚的顶点数量

$V_{hcc}$ - number of vertices describing interaction of $h_{ij}$ with the ghosts

$V_{hcc}$ - 描述 $h_{ij}$ 与鬼场相互作用的顶点数量

$V_{Ncc}$ - number of vertices describing interaction of $N^i$ with the ghosts

$V_{Ncc}$ - 描述 $N^i$ 与鬼场相互作用的顶点数量

$L$ - number of loops, i.e., number of independent loop integrals

$L$ - 圈数，即独立圈积分的数量

$l_N$ - number of external $N$ -legs

$l_N$ - 外部 $N$ 支脚数量

$T$ - number of time derivatives acting on external legs

$T$ - 作用在外部支脚上的时间导数数量

$X$ - number of spatial derivatives acting on external legs

$X$ - 作用在外部支脚上的空间导数数量

These quantities obey two relations:

这些量满足两个关系:

$$L = P_{hh} + P_{NN} + P_{cc} - V_{[h]} - V_{[h]N} - V_{[h]NN} - V_{hcc} - V_{Ncc} + 1, \tag{44}$$

$$l_N = V_{[h]N} + V_{Ncc} + 2V_{[h]NN} - 2P_{NN}. \tag{45}$$

The first relation follows from the standard reasoning that out of $\sum P = P_{hh} + P_{NN} + P_{cc}$ original integrals over frequencies and momenta $\sum V - 1 = V_{[h]} + V_{[h]N} + V_{[h]NN} + V_{hcc} + V_{Ncc} - 1$ of them are removed by the $\delta$ -functions at the vertices (one $\delta$ -function remains as an overall factor multiplying the whole diagram). The second relation is obtained by counting the $N$ -legs. Indeed, each vertex of the type $V_{[h]N}$ or $V_{Ncc}$ brings one $N$ -leg, whereas the vertex $V_{[h]NN}$ brings two; every $\langle N^i N^j \rangle$ -propagator absorbs two legs; the remaining $N$ -legs are external.

第一个关系由标准推导得到: 在对频率和动量的 $\sum P = P_{hh} + P_{NN} + P_{cc}$ 个原始积分中，有 $\sum V - 1 = V_{[h]} + V_{[h]N} + V_{[h]NN} + V_{hcc} + V_{Ncc} - 1$ 个积分被顶点处的 $\delta$ 函数消去 (其中一个 $\delta$ 函数保留，作为整个图的整体乘因子)。第二个关系可通过对 $N$ 支脚计数得到: 每个 $V_{[h]N}$ 型或 $V_{Ncc}$ 型顶点贡献一个 $N$ 支脚，$V_{[h]NN}$ 型顶点贡献两个 $N$ 支脚；每个 $\langle N^i N^j \rangle$ 传播子消耗两个支脚，剩余的 $N$ 支脚即为外部支脚。

The numbers of time and space derivatives in vertices are given in the following table:

顶点中时间导数和空间导数的数量如下表所示:

| Vertex | #of vertex derivatives |
|---|---|
| $V_{[h]}$ | $\partial_x^{2d}$ |
| $V_{[h]N}$ | $\partial_\tau \partial_x$ |
| $V_{[h]NN}$ | $\partial_x^2$ |
| $V_{hc\bar{c}}$ | $\partial_x^{2d}$ |
| $V_{Nc\bar{c}}$ | $\partial_\tau \partial_x$ |

The scaling dimensionality of any block of the full propagator in momentum space is obviously related to dimensionalities of relevant fields in the coordinate space $[\langle \phi_1(p)\phi_2(-p)\rangle]_p = [\phi_1]_x + [\phi_2]_x - 2d$ , so that from the dimensions of fields in $x$ -space,

动量空间中完整传播子任意区块的标度维数显然与坐标空间中相关场的维数有关 $[\langle\phi_1(p)\phi_2(-p)\rangle]_p = [\phi_1]_x + [\phi_2]_x - 2d$，因此根据 $x$ 空间中场的维数,

$$[h]_x = 0,\ [N]_x = d-1,\ [c]_x = [\bar{c}]_x = 0, \tag{46}$$

one finds the scaling dimensions of various blocks of their two-point momentum-space propagators,

即可得到两点动量空间传播子各区块的标度维数,

$$[\langle hh\rangle]_p = -2d,\ [\langle NN\rangle]_p = -2,\ [\langle c\bar{c}\rangle]_p = -2d. \tag{47}$$

Hence, the degree of divergence of any Feynman diagram equals

因此，任意费曼图的发散度等于

$$\begin{aligned} \mathcal{D}_{\text{div}} &= 2dL - 2dP_{hh} - 2P_{NN} - 2dP_{c\bar{c}} \\ &+2dV_{[h]} + (d+1)V_{[h]N} + 2V_{[h]NN} + 2dV_{hc\bar{c}} + (d+1)V_{Nc\bar{c}} \\ &-dT - X = 2d - dT - X - (d-1)\ell_N, \end{aligned} \tag{48}$$

where the first line is contributed by loop momenta integration measure and the full set of propagators, the second line is a contribution of vertex scalings, and the third line corresponds to the reduction of the total scaling of degree of divergence due to time and space derivatives on the external lines.

其中第一行来自圈动量积分测度和全套传播子，第二行是顶点标度的贡献，第三行对应外线上时间和空间导数带来的发散度总标度降低。

Using the above relations (44) and (45) we get a remarkable expression for $\mathcal{D}_{\text{div}}$ which is independent of the internal structure of the diagram,

利用上述关系式 (44) 和 (45)，我们得到了 $\mathcal{D}_{\text{div}}$ 一个与图内部结构无关的出色表达式,

$$\mathcal{D}_{\text{div}} = 2d - dT - X - (d-1)\ell_N. \tag{49}$$

We see that $D_{\text{div}}$ is negative for diagrams with more than 2 time or $2d$ space derivatives on external legs. Therefore, only diagrams with at most 2 time and $2d$ space derivatives on the external lines must be renormalized. The corresponding counterterms are polynomial in external frequencies and momenta and hence local in position space. Again, they have no more than 2 time or $2d$ space derivatives acting on the metric $h_{ij}$. In other words, their scaling dimension is less or equal four. If we further assume that the divergent parts of the diagrams respect the local foliation-preserving diffeomorphisms - this will be discussed in the next section - it follows that the counterterms must have the same form as the terms already present in the action (10), (12). This amounts to renormalizability [40].

我们看到，在外腿带有超过 2 个时间导数或 $2d$ 个空间导数的图中，$D_{\rm div}$ 为负。因此，只有外线上至多带有 2 个时间导数和 $2d$ 个空间导数的图需要重整化。对应的抵消项是外频率和动量的多项式，因此在位置空间是定域的。并且，它们作用在度规 $h_{ij}$ 上的时间导数不超过 2 个，空间导数也不超过 $2d$ 个。换言之，它们的标度维数小于等于四。如果进一步假设图的发散部分满足保持定域叶状结构的微分同胚——这一点我们将在下一节讨论——就可以推出抵消项必须与作用量 (10)(12) 中已存在的项形式相同。这就证明了可重整性 [40]。

## Non-projectable Models

## 非投影模型

Non-projectable Hořava gravity models have extra symmetry $t \mapsto \tilde{t}(t)$ and generic space-inhomogeneous lapse function $N = N(t, \mathbf{x}) \neq 1$ , which leads to extra contributions in the potential term of the action (10). For illustration of the general situation we take the model in $(2+1)$ -dimensions [29] which is technically much simpler than its $(3+1)$ -dimensional counterpart. In this case the potential contains ten inequivalent terms,

非投影霍拉瓦引力模型具有额外对称性 $t \mapsto \tilde{t}(t)$ 和一般空间非均匀推移函数 $N = N(t, \mathbf{x}) \neq 1$ ，这会给作用量的势项带来额外贡献 (10)。为说明一般情况，我们以 $(2+1)$ 维模型 [29] 为例，它在技术上比 $(3+1)$ 维对应模型简单得多。这种情况下，势包含十个不等价项，

$$\mathcal{V} = 2\Lambda - \eta R - \alpha a_i a^i + \mu R^2 + \rho_1 \Delta R + \rho_2 R a_i a^i$$

$$+\rho_3 \left(a_i a^i\right)^2 + \rho_4 a_i a^i \nabla_j a^j + \rho_5 \left(\nabla_j a^j\right)^2 + \rho_6 \nabla_i a_j \nabla^i a^j, \tag{50}$$

where

其中

$$a_i = \partial_i \ln N \tag{51}$$

is the ”acceleration” variable which is invariant under the reparameterizations of time; see Eqs. (8). Again tuning the cosmological constant $\Lambda$ to zero and expanding around flat spacetime, one obtains the quadratic action,

是“加速度”变量，它在时间重参数化下保持不变；见式 (8)。再次将宇宙学常数 $\Lambda$ 调为零并在平直时空附近展开，可得二次作用量，

$$S^{(2)}_{d=2,\mathrm{n-p}} = \frac{1}{G} \int \mathrm{d}t\, \mathrm{d}^2 x \bigg[ -\frac{1}{2} (\dot{v}_i - u_i) \Delta (\dot{v}_i - u_i) + \frac{\dot{\psi}^2}{4} + \frac{1}{4} \big(\dot{E} - 2\Delta B\big)^2$$

$$-\frac{\lambda}{4} \big(\dot{\psi} + \dot{E} - 2\Delta B\big)^2 - \mu (\Delta\psi)^2$$

$$-\eta\phi\Delta\psi - \alpha\phi\Delta\phi + \rho_1\phi\Delta^2\psi - (\rho_5 + \rho_6)\phi\Delta^2\phi\big],$$

(52)

where in addition to the projectable case (16) one gets the contribution of the fluctuation of the lapse $\phi \equiv N - 1-$ a new variable devoid of the kinetic term. Exclusion of this variable by its variational equation - the second class constraint - leads to the propagation of a single scalar degree of freedom with the dispersion relation non-polynomial in the momentum,

其中除了投影情况 (16) 的结果，还会得到推移涨落 $\phi \equiv N - 1-$ 的贡献——这是一个没有动能项的新变量。通过变分方程 (第二类约束) 消去该变量后，会得到一个具有动量非多项式色散关系的单标量自由度传播，

$$\omega^2 = \left(\frac{1-\lambda}{1-2\lambda}\right)\frac{\eta^2k^2 + (4\alpha\mu + 2\eta\rho_1)k^4 + \left(\rho_1^2 - 4\mu(\rho_5+\rho_6)\right)k^6}{\alpha - (\rho_5+\rho_6)k^2}. \tag{53}$$

In contrast to the projectable case, this dispersion relation is linear at low $k, \omega^2 = k^2\eta^2(1-\lambda)/\alpha(1-2\lambda)$ , but at large momenta it respects the anisotropic scaling (3),

与投影情况不同，该色散关系在低 $k, \omega^2 = k^2\eta^2(1-\lambda)/\alpha(1-2\lambda)$ 下为线性，但在大动量时满足各向异性标度 (3),

$$\omega^2 = \frac{1-\lambda}{1-2\lambda}\left(4\mu - \frac{\rho_1^2}{\rho_5+\rho_6}\right)k^4. \tag{54}$$

The mode has positive energy and is stable at all momenta for an appropriate choice of parameters $G > 0, \lambda < 1/2$ or $\lambda > 1, \alpha > 0, (\rho_5+\rho_6) < 0$ , and $4\mu > \rho_1^2/(\rho_5+\rho_6)$ .

只要合适选取参数 $G > 0, \lambda < 1/2$ 、$\lambda > 1, \alpha > 0, (\rho_5+\rho_6) < 0$ 和 $4\mu > \rho_1^2/(\rho_5+\rho_6)$ ，该模式能量为正，在所有动量下都是稳定的。

If one considers the UV behavior of the model, which amounts to setting $\eta = \alpha = 0$ , and considers the gauge-fixing procedure with (31) and (32), then the resulting propagators will have irregular terms in their $\langle\phi\phi\rangle$ and $\langle\phi h_{ij}\rangle$ sectors [40]. They cannot be removed by any gauge choice and correspond to the instantaneous interaction present in the theory [17, 27]. We conclude that the correlators of the lapse contain genuinely nonlocal terms violating sufficient conditions of the renormalizability of the theory, derived above. This casts serious doubt on renormalizability of non-projectable Hořava model. However, regularity of propagators is only a sufficient rather than necessary condition of renormalizability, and there might be subtle mechanisms of cancellation of irregular UV divergences caused by these terms.

如果研究该模型的紫外行为 (即令 $\eta = \alpha = 0$ )，并结合 (31) 和 (32) 讨论规范固定过程，那么得到的传播子在 $\langle\phi\phi\rangle$ 和 $\langle\phi h_{ij}\rangle$ 扇区会存在非正则项 [40]。这些项无法通过任何规范选择消除，对应理论中存在的瞬时相互作用 [17, 27]。我们得出结论: 推移函数关联函数中存在真正的非局域项，违反了我们前文推导的理论可重整性充分条件。这让人们对非投影霍拉瓦模型的可重整性产生严重怀疑。不过，传播子正则性只是可重整性的充分而非必要条件，或许存在微妙机制可以抵消这些项带来的紫外反常发散。

The source of these terms is, actually, the presence of second-class constraints in the canonical formalism of non-projectable model [60-63]. It should be emphasized that with this type of constraints direct use of Faddeev-Popov gauge-fixing [64] in Lagrangian formalism is insufficient to produce correct Feynman diagrammatic technique - it should be based on the canonical BFV quantization [65-67] of systems subject to a combination of first-class and second-class constraints. It incorporates nontrivial, time-local $\sim \delta^{(1)}(0)$, path integral measure [68-70], and Dirac brackets in canonical phase space of the theory. Eradication of the second-class constraints by directly solving them and calculation of Dirac brackets generically result in spatial nonlocality of the action which again compromises local renormalization technique. Interestingly, due to peculiarities of the Hořava model this type of nonlocality was circumvented in [45], and, moreover, for the remaining irregular propagator terms the mechanism of their cancellation was observed. In particular, for the (2+1)-dimensional model the power divergent set of contributions of irregular terms $\sim \int d\omega$ were shown to be cancelled by the contribution of the local measure $\sim \delta^{(1)}(0) = \int d\omega/2\pi$. This opens serious prospects for non-projectable Hořava model that could have served as the only known at present candidate for local, unitary, renormalizable gravity theory compatible with general relativity in IR domain [17].

实际上，这些项的来源是非投影模型正则形式中存在第二类约束 [60-63]。需要强调的是，存在这类约束时, 在拉格朗日形式中直接使用法捷耶夫-波波夫规范固定 [64] 无法得到正确的费曼图技术——该理论的正确量子化应当基于同时存在第一类与第二类约束系统的正则 BFV 量子化 [65-67]。这种量子化包含非平凡的时间局域 $\sim \delta^{(1)}(0)$ 路径积分测度 [68-70]，以及理论正则相空间中的狄拉克括号。通过直接求解来消去第二类约束并计算狄拉克括号后，作用量通常会产生空间非定域性，这同样会破坏局域重整化技术。有趣的是，由于霍拉瓦模型的特殊性，文献 [45] 规避了这类非定域性，并且还发现剩余不规则传播子项存在抵消机制。具体而言，对于 (2+1) 维模型，不规则项 $\sim \int d\omega$ 的幂次发散贡献组可被局域测度 $\sim \delta^{(1)}(0) = \int d\omega/2\pi$ 的贡献抵消。这为非投影霍拉瓦模型打开了可观的研究前景，该模型目前是已知唯一满足条件的候选理论: 它可以成为在红外区与广义相对论相容的局域、幺正、可重整引力理论 [17]。

## BRST Structure of Renormalization and Covariance of Counterterms

## 重整化的 BRST 结构与抵消项的协变性

We also must provide the gauge invariance of the counterterms. In the perturbative expansion around flat spacetime, considered so far, gauge invariance is actually not preserved. The way to proceed would be to exploit the BRST symmetry of the gauge-fixed action to constrain the structure of counterterms, similar to the analysis of [2]. Even more efficient is to adopt the background field formalism and the method of background-covariant gauges [32-34] where the gauge invariance becomes manifest.

我们还必须保证抵消项的规范不变性。在本文目前所讨论的平坦时空微扰展开中，规范不变性实际上并未得到保留。和文献 [2] 的分析类似，我们可以利用规范固定作用量的 BRST 对称性来约束抵消项的结构。更高效的方法是采用背景场形式论与背景协变规范法 [32-34]，该方法中规范不变性是显然成立的。

BRST structure of the renormalization in such gauges, which is supposed to lead to covariant counterterms and reduce to the renormalization of the coupling constants in the original action (10), requires extension of known results for Lorentz-invariant theories to Hořava gravity. Such an extension is possible [41] within a class of theories with a generic closed algebra of irreducible gauge generators which are linear in the quantum fields $\varphi$ . This extension runs via the inclusion into the conventional BRST operator $Q$ the background field $\phi$ and its BRST partner along with a special choice of the gauge fermion $\Psi_{\text{ext}}[\Phi, \Phi^*]$ which depends on the full set of quantum fields $\Phi = (\varphi, c, \bar{c}, b)$ (including together with $\varphi$ the BRST ghosts $c, \bar{c}$ and Lagrange multipliers $b$ ) and their antifields $\Phi^*$ , the latter playing the role of the sources of the BRST transformations of the full set of $\Phi$ ,

这类规范下重整化的 BRST 结构应当能导出协变抵消项，并退化为原作用量 (10) 中耦合常数的重整化，该结构需要将洛伦兹不变理论的已知结论推广到霍拉瓦引力。对于一类由不可约规范生成元构成一般封闭代数、且生成元对量子场 $\varphi$ 呈线性的理论，这类推广是可行的 [41]。该推广的实现方式是: 在传统 BRST 算符 $Q$ 中引入背景场 $\phi$ 及其 BRST 伙伴，并对规范费米子 $\Psi_{\text{ext}}[\Phi, \Phi^*]$ 做特殊选取——规范费米子依赖于全套量子场 $\Phi = (\varphi, c, \bar{c}, b)$ (除 $\varphi$ 外还包括 BRST 鬼 $c, \bar{c}$ 和拉格朗日乘子 $b$ ) 及其反场 $\Phi^*$ ，其中反场承担全套 $\Phi$ BRST 变换源的作用，

$$Q \Rightarrow Q_{\text{ext}}\,, \ \Psi \Rightarrow \Psi_{\text{ext}}[\Phi, \Phi^*]\,. \tag{55}$$

The resulting generating functional $W[J, \Phi^*]$ , the functional of the sources $J$ of the quantum fields $\Phi$ and their antifields $\Phi^*$ , which is given by the path integral

由此得到的生成泛函 $W[J, \Phi^*]$ ，即量子场 $\Phi$ 及其反场 $\Phi^*$ 的源 $J$ 的泛函，由路径积分给出

$$\exp\left(-\frac{1}{\hbar} W[J, \Phi^*]\right) = \int D\Phi e^{-(S[\varphi]+Q_{\text{ext}}\Psi_{\text{ext}}[\Phi,\Phi^*]+J\Phi)/\hbar}, \tag{56}$$

satisfies well-known Slavnov-Taylor identities and also solves special Ward identities. The latter hold provided the gauge fermion is built of the so-called background-covariant gauge conditions which make the gauge fermion invariant under the simultaneous gauge transformations of both the quantum field $\varphi$ and its background counterpart $\phi$ .

满足著名的斯拉诺夫-泰勒恒等式，也满足特殊的沃德恒等式。后者成立的条件是，规范费米子由所谓背景协变规范条件构造，这类条件使得规范费米子在量子场 $\varphi$ 及其背景对应量 $\phi$ 同时做规范变换下保持不变。

The application of Slavnov-Taylor and Ward identities to the divergent part of the effective action runs perturbatively in $\hbar$ via the study of the cohomologies of the nilpotent BRST operator $Q_{\text{ext}}$ . It yields the needed BRST structure - the overall effect reduces to a simultaneous local renormalization of the action $S[\varphi]$ by gauge-invariant counterterms of the original gauge fields $\varphi$ and the renormalization of the gauge fermion $\Psi_{\text{ext}}[\Phi, \Phi^*]$ which also gets quantum corrections,

将斯拉诺夫-泰勒恒等式与沃德恒等式应用于有效作用量的发散部分时，需要在 $\hbar$ 框架下微扰地研究幂零 BRST 算符 $Q_{\text{ext}}$ 的上同调。这可以得到我们需要的 BRST 结构: 整体效应等价于用原规范场 $\varphi$ 的规范不变抵消项对作用量 $S[\varphi]$ 同时做局域重整化，同时重整化规范费米子 $\Psi_{\text{ext}}[\Phi, \Phi^*]$ ——后者也会得到量子修正,

$$S[\varphi] \to S[\varphi] + \Delta_\infty S[\varphi] \tag{57}$$

$$\Psi_{\text{ext}}\,[\Phi,\Phi^*] \to \boldsymbol{\Psi}_{\text{ext}}\,[\Phi,\Phi^*] + \Delta_\infty \boldsymbol{\Psi}_{\text{ext}}\,[\Phi,\Phi^*]\,. \tag{58}$$

This BRST structure of renormalization is achieved by means of additional renormalization of quantum fields, which turns out to be an (generically nonlinear) anti-canonical transformation generated by the gauge fermion $\Psi_{\text{ext}}$ itself [41]. It is important that the uncontrollably complicated renormalization of the gauge fermion, indiscriminately depending on all quantum fields $\Phi$ , is immaterial from the viewpoint of physical applications, because anyway the generating functional of physical amplitudes is gauge-independent onshell, $\delta_\Psi W|_{J=\Phi^*=0} = 0$ . A remarkable feature of this scheme is that it applies not only to perturbatively renormalizable theories but also to effective field theories below their cutoff [41]. All this justifies the physically invariant scope of these results and their applications, in particular, to Hořava gravity models.

这种重整化的 BRST 结构通过量子场的额外重整化实现，而该额外重整化本质上是规范费米子 $\Psi_{\text{ext}}$ 本身诱导的 (一般为非线性) 反正则变换 [41]。重点在于，规范费米子不受控的复杂重整化 (无差别地依赖所有量子场 $\Phi$ ) 对物理应用而言无关紧要，因为物理振幅的生成泛函在壳上本来就是规范无关的， $\delta_\Psi W|_{J=\Phi^*=0} = 0$ 。该框架的一个显著特点是，它不仅适用于可微扰重整化的理论，也适用于截止以下的有效场论 [41]。所有这些都证明了这些结论及其应用 (尤其应用于霍拉瓦引力模型时) 在物理上是自洽合理的。

In the context of Hořava gravity the construction of background-covariant gauges starts with the decomposition of the full set of gauge fields $\varphi = (\gamma_{ij}, N^i)$ into their background $\phi = (g_{ij}, \mathcal{N}^i)$ and quantum fluctuations $(h_{ij}, n^i)$ :

在霍拉瓦引力的框架下，背景协变规范的构造从将全套规范场 $\varphi = (\gamma_{ij}, N^i)$ 分解为背景部分 $\phi = (g_{ij}, \mathcal{N}^i)$ 和量子涨落 $(h_{ij}, n^i)$ 开始:

$$\gamma_{ij} = g_{ij} + h_{ij},\ N^i = \mathcal{N}^i + n^i. \tag{59}$$

The background-covariant gauge conditions for these fluctuations are just the covariantization of all formulas of the previous sections. Instead of (31) and (32) we write

这些涨落的背景协变规范条件正是前述各节所有公式的协变化。我们将 (31) 和 (32) 改写为

$$F^i = D_t n^i + \frac{1}{2\sigma}\left(O^{-1}\right)^{ij}\left(D_k h^k_j - \frac{\lambda}{2\sigma} D_j h\right), \tag{60}$$

$$O_{ij} = \left[g^{ij}(-\Delta)^{d-1} - \xi D^i (-\Delta)^{d-2} D^j\right]^{-1}. \tag{61}$$

where

其中

$$D_t n^i = \dot{n}^i - \mathcal{N}^k D_k n^i + n^k D_k \mathcal{N}^i \tag{62}$$

is the covariant time derivative and $D_i$ are the covariant derivatives conserving the background metric $g_{ij}, \Delta = g^{ij} D_i D_j$ is the respective covariant Laplacian, all indices are raised and lowered by $g^{ij}$ and $g_{ij}$ , and

$h = h_{ij}g^{ij}$ . Lack of commutativity of $D_i$ explains the operator ordering in the definition (61) of the symmetric operator $O_{ij}$ . The gauge-fixing action is still given by the Lagrangian (30) that must be integrated over the spacetime with the covariant measure $\int d\tau d^d x\sqrt{g}$ , $g = \det g_{ij}$ . Finally, for the gauge transformations (39)-(40) their covariantization reduces to identically rewriting them in the form

为协变时间导数，$D_i$ 是保持背景度量不变的协变导数，$g_{ij}, \Delta = g^{ij}D_iD_j$ 是对应的协变拉普拉斯算符，所有指标都由 $g^{ij}$ 和 $g_{ij}$ 、$h = h_{ij}g^{ij}$ 升降。$D_i$ 不对易性解释了对称算符 $O_{ij}$ 定义式 (61) 中的算符排序。规范固定作用量仍由拉格朗日量 (30) 给出，该拉格朗日量需要在带协变测度 $\int d\tau d^d x\sqrt{g}$ 、$g = \det g_{ij}$ 的时空上积分。最后，对于规范变换 (39)-(40)，它们的协变化可直接将它们改写为如下形式

$$\delta^f h_{ij} = D_i f_j + D_j f_i,\ f_i = g_{ij} f^j, \tag{63}$$

$$\delta^f n^i = \dot{f}^i + f^j D_j n^i - n^j D_j f^i. \tag{64}$$

One should be worried at this point that the gauge-fixing Lagrangian depends on the background fields in a nonlocal manner which can compromise the locality of counterterms (see the remark after Eq. (33)). To resolve this issue, we observe that the nonlocal operator $\mathcal{O}_{ij}$ actually cancels everywhere in the gauge-fixing action, except the kinetic term for the shift,

此时人们会担忧: 规范固定拉格朗日量非局域地依赖背景场，这会破坏抵消项的局域性 (参见式 (33) 后的注记)。为解决该问题，我们发现非局域算符 $\mathcal{O}_{ij}$ 实际上会在规范固定作用量的各处抵消，仅位移场的动能项除外，

$$S_{n,kin}[n] = \frac{\sigma}{2G}\int d\tau d^d x\sqrt{g} D_t n^i O_{ij} D_t n^j. \tag{65}$$

The latter is cast in the local form by introducing an auxiliary field $\pi_i$ ,

通过引入辅助场 $\pi_i$ ，可将后者改写为局域形式，

$$S'_{n,\,\mathrm{kin}}[\pi, n] = \frac{1}{G}\int d\tau d^d x\sqrt{g}\left[\frac{1}{2\sigma}\pi_i\left(O^{-1}\right)^{ij}\pi_j - i\pi_i D_t n^i\right]. \tag{66}$$

Taking the Gaussian path integral over $\pi_i$ reproduces (65). Note that we have introduced an imaginary coefficient in front of the second term in (66) in order to preserve the positivity of the quadratic term (Strictly speaking, this argument applies when the operator $O_{ij}$ is positive-definite, but a possible lack of positivity does not affect the perturbative considerations.). Note that $\pi_i$ enters in the action as a canonically conjugate momentum for the shift perturbations $n^i$ . From this perspective, the presence of an imaginary part in (66) is not surprising, because the imaginary part associated with the canonical form always appears when the Euclidean action is written in terms of canonical variables.

对 $\pi_i$ 做高斯路径积分即可得到 (65)。注意我们在 (66) 的第二项前引入虚系数，是为了保证二次项为正 (严格来说，该论证仅在算符 $O_{ij}$ 正定的时候成立，但即便不正定也不影响微扰分析。)。注意 $\pi_i$ 作为正则共轭动量进入作用量，对应位移涨落 $n^i$ 。从该角度看，(66) 中存在虚部并不意外，因为当欧几里得作用量用正则变量表示时，与正则形式关联的虚部总会出现。

It is instructive to work out how the introduction of $\pi_i$ affects the measure in the path integral. Let us make a step backward and recall that the gauge-fixing Lagrangian (30) arises as a result of smearing the delta-function type gauge-fixing condition $F^i = f^i$ with the weighting functional

推导引入 $\pi_i$ 对路径积分测度的影响很有启发性。我们退一步回顾: 规范固定拉格朗日量 (30) 源于用权重泛函对 δ 函数型规范固定条件 $F^i = f^i$ 做弥散,

$$\left(\operatorname{Det} O_{ij}\right)^{1/2} \int Df \exp\left[-\frac{\sigma}{2G} \int \mathrm{d}\tau \mathrm{d}^d x \sqrt{g} f^i O_{ij} f^j\right] \tag{67}$$

inserted in the partition function of the theory. Notice the square root of the functional determinant of the operator $\mathcal{O}_{ij}$ in the prefactor which ensures the correct normalization. Thus, before introducing $\pi_i$ the partition function has the form,

将其插入理论的配分函数中。注意前置因子中算符 $\mathcal{O}_{ij}$ 泛函行列式的平方根，它保证了归一化正确。因此，引入 $\pi_i$ 前，配分函数形式为

$$Z = \left(\operatorname{Det} O_{ij}\right)^{1/2} \int DnDhDcD\bar{c} \exp\left[-\left(S_{n,kin} + \ldots\right)\right], \tag{68}$$

where ellipsis stands for the local contributions in the action. The introduction of $\pi_i$ not only makes the action local, but also absorbs the determinant from the prefactor, which follows from the relation

其中省略号代表作用量中的局域贡献。引入 $\pi_i$ 不仅让作用量成为局域量，还吸收了前置因子中的行列式，这可由下式推出

$$e^{-S_{n,\,\mathrm{kin}}\left[n^i\right]} = \left(\operatorname{Det} O_{ij}\right)^{-1/2} \int D\pi e^{-S'_{n,\,\mathrm{kin}}\left[\pi_j, n^i\right]},$$

so that the final expression for the partition function reads

因此配分函数的最终表达式为

$$Z = \int D\pi DnDhDcD\bar{c} \exp\left[-\left(S'_{n,\,\mathrm{kin}} + \ldots\right)\right]. \tag{69}$$

Curiously, the introduction of $\pi_i$ makes the integration measure in the path integral flat (Liouville like), which further supports the identification of $\pi_i$ as the canonically conjugate momentum to $n^i$ .

有意思的是，引入 $\pi_i$ 后，路径积分的测度变成了平坦测度 (刘维尔型)，这进一步支持 $\pi_i$ 是 $n^i$ 的正则共轭动量这一结论。

Finally, we have to check that the introduction of $\pi_i$ does not spoil the regular structure of the propagators. This is easy to see from the fact that additional blocks of the full propagator $\langle \pi_i(p) n^j(-p) \rangle$ and $\langle \pi_i(p) \pi_j(-p) \rangle$ are also regular and compatible with the scaling dimension $[\pi_i] = 1$ . As a consequence, the reasoning of previous sections remains true with the field $\pi_i$ included into consideration.

最后，我们需要验证引入 $\pi_i$ 不会破坏传播子的正则结构。这一点很容易看出: 全传播子额外的分块 $\langle \pi_i(p) n^j(-p) \rangle$ 和 $\langle \pi_i(p) \pi_j(-p) \rangle$ 也是正则的，且符合标度维度 $[\pi_i] = 1$ 。因此，前述各节的论证在纳入 $\pi_i$ 后仍然成立。

## Asymptotic Freedom in (2+1)-Dimensional Hořava Gravity

## (2+1) 维霍拉瓦引力中的渐近自由

Here we show that $(2+1)$ -dimensional Hořava gravity is asymptotically free in UV limit [42]. Its action in the UV domain,

我们在此证明，$(2+1)$ 维霍拉瓦引力在紫外极限下具有渐近自由 [42]。其作用量在紫外区域

$$S = \frac{1}{2G} \int dt d^2 x \sqrt{\gamma} \left( K_{ij}^2 - \lambda K^2 + \mu R^2 \right), \tag{70}$$

includes three coupling constants $G, \lambda$ , and $\mu$ . However, only two of their combinations are essential [72] in the sense that their renormalization does not depend on the choice of the gauge. The gauge variation induces the transformation of the effective action by terms vanishing on shell, that is, on effective equations of motion of the theory [32, 73]. Since in background-covariant gauges the UV divergent part is local and gauge invariant, such a change in the one-loop order can only be of the form

包含三个耦合常数 $G, \lambda$ 和 $\mu$ 。但它们只有两个组合是本质的 [72]，因为它们的重整化不依赖于规范选择。规范变分诱导有效作用量发生变换，变换项在壳上即该理论的有效运动方程上为零 [32, 73]。由于在背景协变规范中，紫外发散部分是定域且规范不变的，因此单圈阶的这类变换只能具有如下形式

$$\begin{aligned} \Gamma_{1\text{-loop}}^{\text{div}} &\to \Gamma_{1\text{-loop}}^{\text{div}} + \varepsilon \int \mathrm{d}\tau d^2 x \frac{\delta S}{\delta \gamma_{ij}} \gamma_{ij} \\ &= \Gamma_{1\text{-loop}}^{\text{div}} + \frac{\varepsilon}{2G} \int dt d^2 x \sqrt{\gamma} \left( K_{ij}^2 - \lambda K^2 - \mu R^2 \right), \end{aligned} \tag{71}$$

where $\varepsilon$ parameterizes the gauge variation. This is because other combinations of equations of motion are either noncovariant or do not have a needed dimension (Simple derivation of the equality here follows from the fact that under the metric rescaling by a global parameter $a, \gamma_{ij} \to a\gamma_{ij}$ , the kinetic term of the action gets rescaled linearly in $a$ , while the potential term gets rescaled by $1/a$ .). Such a variation of the UV counterterm implies the following change in the coupling constants of the theory:

其中 $\varepsilon$ 是规范变分的参数。这是因为运动方程的其他组合要么是非协变的，要么不满足所需的量纲 (此处等式的简单推导源于以下事实: 当度量整体参数 $a, \gamma_{ij} \to a\gamma_{ij}$ 标度时，作用量的动力学项随 $a$ 线性标度，而势能项则随 $1/a$ 标度)。紫外抵消项的这类变分意味着该理论的耦合常数发生如下变化:

$$\delta_\varepsilon G = -2G^2 \varepsilon, \ \delta_\varepsilon \lambda = 0, \ \delta_\varepsilon \mu = -4G\mu\varepsilon, \tag{72}$$

and implies that only two of their combinations are gauge independent. One of them is the original coupling $\lambda$ and another one is

由此可知，它们只有两个组合是规范无关的。其中一个是原始耦合 $\lambda$，另一个为

$$\mathcal{G} \equiv \frac{G}{\sqrt{\mu}},\ \delta_\varepsilon \mathcal{G} = 0. \tag{73}$$

To perform renormalization of the theory we use the background field formalism of the previous section and calculate the divergent part of the one-loop effective action $\Gamma^{\text{div}}_{1\text{-loop}}\left[g_{ij}, \mathcal{N}^i\right]$ at zero background shift functions $\mathcal{N}^i = 0$ and at the background metric $g_{ij} = \delta_{ij} + H_{ij}$, which is close to flat space, by perturbations in $H_{ij}$. Due to the invariance of the effective action, its divergent part has the structure of the classical action (70) whose expansion begins with the terms quadratic in $H_{ij}$. The UV counterterms and relevant $\beta$-functions are then found by studying how the two-point functions of $H_{ij}$ - the coefficients of expansion of the effective action in $H_{ij}$ - are renormalized after integrating out the quantum fluctuations $h_{ij}$ and $n^i$ (note that $h_{ij}$ and $n^i$ are quantum fluctuations on top of the perturbed background that should be discerned from background perturbations). The renormalization of $G$ is then extracted from the terms $\dot{H}^2_{ij} \sim K^2_{ij}$, while one of $\lambda$ comes from $\dot{H}^2 \sim K^2$. For the renormalization of $\mu$ we can use any of the three structures $\partial_i\partial_j H^{ij}\Delta H$, $\left(\partial_i\partial_j H^{ij}\right)^2$, or $(\Delta H)^2$ contributing to the $R^2$-potential of (70). This explains why we do not need the effective action at nonzero $\mathcal{N}^i$ or the diagrams with external $\mathcal{N}^i$-lines.

为了对该理论进行重整化，我们使用上一节的背景场形式，在零背景位移函数 $\mathcal{N}^i = 0$、背景度量 $g_{ij} = \delta_{ij} + H_{ij}$ (近似平直空间) 下，按 $H_{ij}$ 微扰计算单圈有效作用量 $\Gamma^{\text{div}}_{1\text{-loop}}\left[g_{ij}, \mathcal{N}^i\right]$ 的发散部分。由于有效作用量的不变性，其发散部分具有经典作用量 (70) 的结构，该展开以 $H_{ij}$ 的二次项作为起始项。通过研究积分掉量子涨落 $h_{ij}$ 和 $n^i$ 后，两点函数 (有效作用量按 $H_{ij}$ 展开的系数，即 $H_{ij}$ 的两点函数) 的重整化，我们得到了紫外抵消项和相关的 $\beta$ 函数 (请注意，$h_{ij}$ 和 $n^i$ 是受扰背景上的量子涨落，需要与背景扰动区分开)。之后，$G$ 的重整化可从项 $\dot{H}^2_{ij} \sim K^2_{ij}$ 中提取，而 $\lambda$ 之一可从 $\dot{H}^2 \sim K^2$ 得到。对于 $\mu$ 的重整化，我们可以使用对 (70) 的 $R^2$ 势能有贡献的三个结构 $\partial_i\partial_j H^{ij}\Delta H$、$\left(\partial_i\partial_j H^{ij}\right)^2$ 或 $(\Delta H)^2$ 中的任意一个。这解释了为什么我们不需要非零 $\mathcal{N}^i$ 处的有效作用量，也不需要带外 $\mathcal{N}^i$ 线的费曼图。

The propagators of quantum fields $h_{ij}$ and $n_i$ considered above, the momentum $\pi_i$ (introduced in (66)), and the ghosts $\bar{C}^i$ and $C^i$ are particularly simple in the gauge

上文讨论的量子场 $h_{ij}$ 和 $n_i$ 的传播子、动量 $\pi_i$ (式 (66) 中引入) 以及鬼场 $\bar{C}^i$ 和 $C^i$ 在该规范中形式格外简单:

$$\sigma = \frac{1-2\lambda}{8\mu\,(1-\lambda)},\ \xi = -\frac{1-2\lambda}{2\,(1-\lambda)}, \tag{74}$$

where they are all proportional to the propagator of the physical scalar mode (see Eq. (18) for the dispersion relation of $d = 2$ model in Lorentzian spacetime),

在该规范下，它们都正比于物理标量模式的传播子 (洛伦兹时空下 $d = 2$ 模型的色散关系见式 (18)),

$$\mathcal{P}_{\text{s}}\,(\omega, p) = \left[\omega^2 + 4\mu\frac{1-\lambda}{1-2\lambda}\,p^4\right]^{-1}. \tag{75}$$

The vertices required for the one-loop calculation can be found by expanding the total gauge-fixed action up to second order in the background field $H_{ij}$ . The diagrams which give rise to logarithmic divergences are shown in Fig. 1.

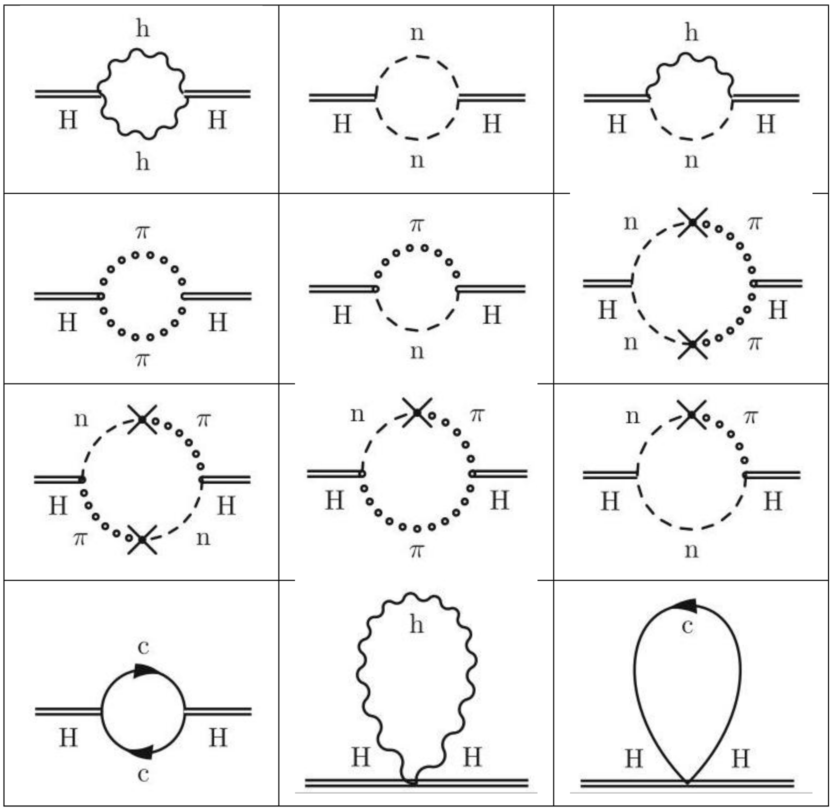
单圈计算所需顶点可通过将总规范固定作用量在背景场 $H_{ij}$ 展开至二阶得到。产生对数发散的费曼图如图 1 所示。

The computation is simplified by considering the renormalization of $\{G, \lambda\}$ and of $\mu$ separately. This can be done by evaluating the quadratic part of the effective action $\Gamma\left[\delta_{ij}+H_{ij}\right]$ on time- or space-dependent backgrounds which correspond respectively to diagrams with vanishing spatial momenta or frequency in external legs. Thus, $\{G, \lambda\}$ -renormalization follows from the logarithmically divergent diagrams carrying only external frequency $\Omega$ at vanishing external momentum $P_i$ , whereas $\mu$ -renormalization originates from those with only external momentum at vanishing $\Omega$ .

分别考虑 $\{G, \lambda\}$ 和 $\mu$ 的重整化可简化计算。我们可以通过在分别对应外腿空间动量为零或频率为零的图的含时或含空间背景上计算有效作用量 $\Gamma\left[\delta_{ij}+H_{ij}\right]$ 的二次项来完成该分离。因此，$\{G, \lambda\}$ 重整化来源于外动量 $P_i$ 为零、仅带外频率 $\Omega$ 的对数发散图，而 $\mu$ 重整化来源于 $\Omega$ 为零、仅带外动量的对数发散图。

Fig. 1 Feynman diagrams (bubbles and fishes) for the two-point function of $H_{ij}$ . The cross represents the mixed propagator $\langle n^i \pi^j\rangle$

图 1 $H_{ij}$ 两点函数的费曼图 (泡图和鱼图)。叉号代表混合传播子 $\langle n^i \pi^j\rangle$




The typical loop integral over internal momentum and frequency has the form,

对内部动量和频率的典型圈积分具有如下形式:

$$\int \frac{\mathrm{d}\omega \mathrm{d}^2 q}{(2\pi)^3} \omega^{2a} q^{2b} \prod_I \mathcal{P}_s(\omega + \Omega_I, q + P_I), \tag{76}$$

with constant $a, b$ , and $\{\Omega_I, P_I\}$ - the relevant external frequencies and two momenta. Logarithmically divergent contributions proportional to $\Omega^2$ or $P^4$ , which renormalize the terms of the bare action (70), follow from the Taylor expansion of the integrand of (76) up to the desired order in external frequency or momentum, such that the final integrands all acquire the general form

其中 $a, b$ 是常数，$\{\Omega_I, P_I\}$ 为相关外频率和两个外动量。从 (76) 被积函数对外部频率或动量展开至所需阶数，即可得到正比于 $\Omega^2$ 或 $P^4$ 的对数发散贡献，这些贡献重整化裸作用量 (70) 中的项，使得最终被积函数都具有如下一般形式

$$\mathcal{I}[a, b, A] = \omega^{2a} q^{2b} (\mathcal{P}_s(\omega, q))^A = \frac{\omega^{2a} q^{2b}}{\Gamma(A)} \int_0^\infty \mathrm{d}s s^{A-1} e^{-s(\mathcal{P}_s(\omega, q))^{-1}}, \tag{77}$$

with $A$ being a constant power. The integral over frequency and momentum in (76) can then be expressed in terms of the $\Gamma$ -functions. In this "proper time" representation logarithmic UV divergences appear as the integral divergent at $s = 0$ , $\int_0^\infty \mathrm{d}s/s$ , which when regulated by the UV cutoff $\Lambda_{\mathrm{UV}}$ reads as

其中 $A$ 是常幂次。(76) 中对频率和动量的积分可以用 $\Gamma$ 函数表示。在该“固有时间”表示中，紫外对数发散表现为在 $s = 0$ 、$\int_0^\infty \mathrm{d}s/s$ 处发散的积分，用紫外截断 $\Lambda_{\mathrm{UV}}$ 正规化后该积分可写为

$$\int_0^\infty \frac{\mathrm{d}s}{s} \mapsto \log\left(\frac{\Lambda_{\mathrm{UV}}^4}{k_*^4}\right), \tag{78}$$

where $k_*$ is a subtraction point. Here we have taken into account that the proper time parameter $s$ has scaling dimension 4 .

其中 $k_*$ 是减除点。此处我们已经考虑到固有时间参数 $s$ 的标度维数为 4。

As a result the UV finite renormalized coupling constants $G$ and $v_a = \{\lambda, \mu\}$ , $a = 1, 2$ , express in the one-loop approximation in terms of the bare (divergent) couplings $G_0, v_a^0$ via the following equations:

最终，紫外有限的重整化耦合常数 $G$ 、$v_a = \{\lambda, \mu\}$ 和 $a = 1, 2$ 在单圈近似下可通过以下方程用裸 (发散) 耦合 $G_0, v_a^0$ 表示:

$$\frac{1}{2G} = \frac{1}{2G_0} + C_G \ln \frac{\Lambda_{\mathrm{UV}}^2}{k_*^2}, \quad \frac{v_a}{2G} = \frac{v_a^0}{2G_0} + C_{v_a} \ln \frac{\Lambda_{\mathrm{UV}}^2}{k_*^2}, \tag{79}$$

where $C_G$ and $C_{v_a}$ are some independent of $G$ (and $G_0$ ) coefficient functions of $\lambda$ and $\mu$ - a primary goal of one-loop calculations. Then, the full set of beta-functions of all renormalized couplings, defined according to standard rules of renormalization group theory as the derivatives with respect to the running scale $k_*$ , reads

其中 $C_G$ 和 $C_{v_a}$ 是不依赖于 $G$ (和 $G_0$ ) 的系数函数，而 $\lambda$ 和 $\mu$ 是单圈计算的核心目标。根据重整化群理论的标准规则，所有重整化耦合的完整 β 函数定义为对跑动能标 $k_*$ 的导数，其形式为

$$\beta_G \equiv \frac{dG}{d\ln k_*} = 4G^2 C_G,\ \beta_{v_a} \equiv \frac{dv_{a,\,\mathrm{ren}}}{d\ln k_*} = -4GC_{v_a} + v_a \frac{\beta_G}{G}. \tag{80}$$

Calculation of $C_G$ and $C_{v_a}$ then gives [42]

计算 $C_G$ 和 $C_{v_a}$ 后可得文献 [42] 的结果:

$$\beta_\lambda = \frac{15 - 14\lambda}{64\pi}\sqrt{\frac{1-2\lambda}{1-\lambda}}\mathcal{G}, \tag{81}$$

$$\beta_\mu = \frac{30 - 73\lambda + 42\lambda^2}{32\pi(1-\lambda)^{3/2}\sqrt{1-2\lambda}} G\sqrt{\mu}, \beta_G = -\frac{30\lambda - 23}{32\pi\sqrt{(1-2\lambda)(1-\lambda)}}\frac{G^2}{\sqrt{\mu}}. \tag{82}$$

The last two beta-functions separately do not make much sense, because their couplings are not essential and depend on the choice of gauge, but their combination yields the beta-function of essential $\mathcal{G} = G/\sqrt{\mu}$ ,

最后两个 β 函数单独来看没有太多意义，因为它们对应的耦合并非本质耦合，且依赖规范选取，但二者组合可以得到本质耦合 $\mathcal{G} = G/\sqrt{\mu}$ 的 β 函数:

$$\beta_{\mathcal{G}} = -\frac{\left(16 - 33\lambda + 18\lambda^2\right)}{64\pi(1-\lambda)^2}\sqrt{\frac{1-\lambda}{1-2\lambda}}\mathcal{G}^2, \tag{83}$$

which is indeed gauge independent as it can be checked by the calculation in the alternative gauge. Namely, it was checked in the background gauge with $\xi = 0$ (and with $\sigma$ as in (74)) and also outside of the family (60)-(61), in the conformal gauge, $h_{ij} = e^{2\phi}\gamma_{ij}$ , which is possible in two spatial dimensions. In the degenerate (delta-function type) conformal gauge the beta-functions of $G$ and $\mu$ are different from (82),

该 β 函数确实是规范无关的，我们可以通过在其他规范下计算验证这一点: 具体来说，我们已经在 $\xi = 0$ 背景规范 (其中 $\sigma$ 和 (74) 中一致) 下做了验证，同时也在 (60)-(61) 族之外的共形规范 $h_{ij} = e^{2\phi}\gamma_{ij}$ 中做了验证，二维空间中可以选取该规范。在简并 (δ 函数型) 共形规范下，$G$ 和 $\mu$ 的 β 函数与 (82) 不同,

$$\beta_\mu = \frac{2 - 7\lambda + 6\lambda^2}{32\pi(1-\lambda)^{3/2}\sqrt{1-2\lambda}} G\sqrt{\mu},\ \beta_G = -\frac{6\lambda - 7}{32\pi\sqrt{(1-2\lambda)(1-\lambda)}}\frac{G^2}{\sqrt{\mu}},$$

but they lead to the same $\beta_{\mathcal{G}}$ given by (83). Furthermore, it matches with the results of [71].

但它们得到的 $\beta_{\mathcal{G}}$ 和 (83) 给出的结果一致。此外，该结果也和文献 [71] 的结果吻合。

The structure of the RG flow generated by beta-functions in the regions of unitarity, $\lambda < 1/2$ and $\lambda > 1$ , is shown in Fig. 2. The theory possesses two UV fixed points: $(\lambda, \mathcal{G}) = (1/2, 0)$ and $(\lambda, \mathcal{G}) = (15/14, 0)$ . The first fixed point is located at the boundary of the allowed region and corresponds to strong coupling, as is clear from the singularity in (83). However, the structure of $\beta_{\mathcal{G}}$ suggests that the actual expansion parameter in the limit $\lambda \to 1/2$ is $\widetilde{\mathcal{G}} = \mathcal{G}/\sqrt{1-2\lambda}$ , with the $\beta$ -function $\beta_{\widetilde{\mathcal{G}}} = -(1-2\lambda)^2\widetilde{\mathcal{G}}^2/64\pi(1-\lambda)^{3/2}$ . It vanishes at $\lambda \to 1/2$ , so that $\widetilde{\mathcal{G}}$ freezes at a constant value in the UV - at the one-loop level there is a family of UV fixed points parameterized by the asymptotic value of $\widetilde{\mathcal{G}}$ . The status of this fixed-point family can be clarified only by taking into account contributions from higher-order and matter loops.

幺正性区域内由 β 函数生成的 RG 流结构，即 $\lambda < 1/2$ 和 $\lambda > 1$，如图 2 所示。该理论存在两个紫外不动点: $(\lambda, \mathcal{G}) = (1/2, 0)$ 和 $(\lambda, \mathcal{G}) = (15/14, 0)$。第一个不动点位于允许区域的边界，对应强耦合，这一点从式 (83) 的奇异性中可以明确看出。不过，$\beta_{\mathcal{G}}$ 的结构表明，在 $\lambda \to 1/2$ 极限下，实际的展开参数是 $\widetilde{\mathcal{G}} = \mathcal{G}/\sqrt{1-2\lambda}$，对应的 β 函数为 $\beta_{\widetilde{\mathcal{G}}} = -(1-2\lambda)^2 \widetilde{\mathcal{G}}^2 / 64\pi(1-\lambda)^{3/2}$。该参数在 $\lambda \to 1/2$ 处趋近于零，因此 $\widetilde{\mathcal{G}}$ 在紫外会固定为一个常数——在单圈水平上，存在一族由 $\widetilde{\mathcal{G}}$ 渐近值参数化的紫外不动点。这族不动点的性质只有纳入高阶和物质圈的贡献才能厘清。

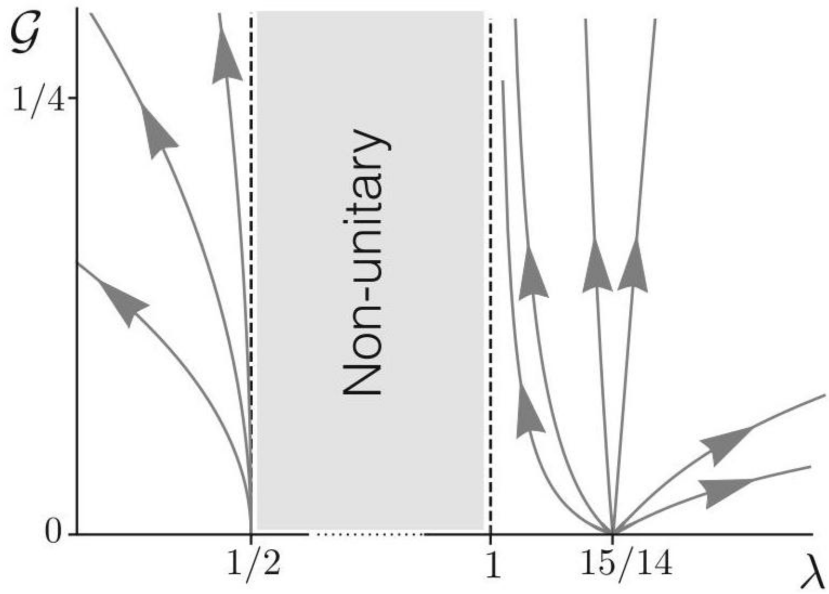



Fig. 2 RG flow of the couplings in

图 2 耦合常数的 RG 流，出自

(2 + 1)-dimensional Hořava gravity. The arrows show the direction of the flow towards the infrared

(2+1) 维霍拉瓦引力。箭头表示流向红外的方向

The second UV fixed point $(\lambda, \mathcal{G}) = (15/14, 0)$ is regular and asymptotically free. In the infrared (IR), the RG trajectories either go to $\lambda \to +\infty, \mathcal{G} \to +\infty$, or $\lambda \to 1^+, \mathcal{G} \to +\infty$. The latter behavior naively corresponds to the relativistic limit of the theory. However, to conclude whether the theory really flows or not to GR requires a non-perturbative analysis as in the IR the system enters into the strong-coupling regime, which is typical for asymptotically free theories. Thus, the RG flow possesses an asymptotically free fixed point in the UV, which establishes this model as a $2+1$ dimensional perturbatively UV-complete theory with a nontrivial propagating gravitational degree of freedom.

第二个紫外不动点 $(\lambda, \mathcal{G}) = (15/14, 0)$ 是正则的，且具有渐近自由性质。在红外 (IR)，RG 轨迹要么流向 $\lambda \to +\infty, \mathcal{G} \to +\infty$，要么流向 $\lambda \to 1^+, \mathcal{G} \to +\infty$。后一种情况从表面上看对应理论的相对论极限。不过，要确定理论是否真的流向广义相对论，还需要进行非微扰分析，因为系统在红外会进入强耦合区域，这是渐近自由理论的典型特征。综上，该 RG 流在紫外存在一个渐近自由不动点，这表明该模型是一个 (2+1) 维的、微扰意义下紫外完备的理论，拥有非平庸的传播引力自由度。

## One-Loop Beta-Functions of (3+1)-Dimensional Hořava Gravity

## (3+1) 维霍拉瓦引力的单圈 β 函数

Physically the most interesting is, of course, the case of $(3+1)$-dimensional theory. Its UV behavior is determined by the action

当然，物理上最令人关注的就是 (3+1) 维理论的情形。它的紫外行为由作用量决定

$$S = \frac{1}{2G} \int d\tau d^3x \sqrt{\gamma} \Big( K_{ij}K^{ij} - \lambda K^2 + \upsilon_1 R^3 + \upsilon_2 R R_{ij} R^{ij} + \upsilon_3 R^i_j R^j_k R^k_i$$

$$+ \upsilon_4 \nabla_i R \nabla^i R + \upsilon_5 \nabla_i R_{jk} \nabla^i R^{jk} \Big), \tag{84}$$

where we retain only its marginal operators. As in $(2+1)$ -dimensional theory, here only a subset of combinations of seven coupling constants, $(G, \lambda, \upsilon_a)$ , $a = 1, \dots 5$ , forms essential couplings.

其中我们仅保留了边缘算符。和 $(2+1)$ 维理论一样，此处七个耦合常数 $(G, \lambda, \upsilon_a)$ , $a = 1, \dots 5$ 仅存在一组组合构成本质耦合。

Similarly to (71) the $\varepsilon$ -variation of the gauge induces the shift of the one-loop divergent part of the effective action by the integrated trace of $\gamma_{ij}$ -equations of motion, corresponding to the rescaling of the three metrics by a global parameter, $\gamma_{ij} \rightarrow a\gamma_{ij}$ . Under this rescaling kinetic and potential terms of the action scale respectively as $a^{\pm 3/2}$ , so that the variation of the one-loop counterterm is again proportional to the difference of the kinetic and potential terms of the classical Lagrangian [43]. This means that the renormalized couplings vary under the change of the gauge as $\delta_\varepsilon G = -2G^2\varepsilon, \delta_\varepsilon \lambda = 0$ , and $\delta_\varepsilon \upsilon_a = -4G\upsilon_a\varepsilon$ , whence the set of gauge independent couplings can be chosen as $\mathcal{G} = G/\sqrt{\upsilon_5}, \lambda$ and $\upsilon_a/\upsilon_5$ . More useful choice is

与 (71) 类似，规范的 $\varepsilon$ 变分会导致有效作用量的单圈发散部分发生偏移，该偏移对应 $\gamma_{ij}$ 运动方程的积分迹，对应三个度规经全局参数 $\gamma_{ij} \rightarrow a\gamma_{ij}$ 重整标度。在该重整标度下，作用量的动能项和势能项分别按 $a^{\pm 3/2}$ 标度，因此单圈抵消项的变分仍正比于经典拉格朗日动能项与势能项之差 [43]。这意味着重整化耦合常数随规范变化按 $\delta_\varepsilon G = -2G^2\varepsilon, \delta_\varepsilon \lambda = 0$ 和 $\delta_\varepsilon \upsilon_a = -4G\upsilon_a\varepsilon$ 变化，因此可以选择 $\mathcal{G} = G/\sqrt{\upsilon_5}, \lambda$ 和 $\upsilon_a/\upsilon_5$ 作为规范无关耦合集合。更适用的选择是

$$\mathcal{G} = \frac{G}{\sqrt{\upsilon_5}},\ \lambda,\ u_s = \sqrt{\frac{1-\lambda}{1-3\lambda}\left(\frac{8\upsilon_4}{\upsilon_5} + 3\right)},\ \upsilon_a = \frac{\upsilon_a}{\upsilon_5},\ a = 1, 2, 3,$$

(85)

In contrast to the $(2+1)$ -dimensional model the level of computational complexity in $(3+1)$ -dimensional theory is so high that the final result cannot be attained by the usual Feynman diagrammatic technique. Only the renormalization of $G$ and $\lambda$ was possible by directly calculating the diagrams [43]. This was done by the renormalization of the $\langle N^i N^j \rangle$ propagator on flat space background, that is, by finding the divergences of one-loop diagrams with two external $\mathcal{N}^i$ -legs in contrast to the diagrams with external $H_{ij}$ -legs used in section ”Asymptotic Freedom in (2+1)-Dimensional Hořava Gravity.” For the renormalization of the rest of coupling constants one should use a more efficient method based on Schwinger-DeWitt [1,46-50] or Gilkey-Seeley [74-76] heat kernel technique and its extension in the form of the method of universal functional traces [48, 49].

The latter, along with a special dimensional reduction transform, turns out to be necessary in order to circumvent the problem of nonminimal and higher-derivative operators. Application of these methods runs as follows.

与 $(2+1)$ 维模型不同，$(3+1)$ 维理论的计算复杂度极高，无法通过常规费曼图技术得到最终结果。仅 $G$ 和 $\lambda$ 的重整化可以通过直接计算图得到 [43]。该计算是通过对平直空间背景下的 $\langle N^i N^j \rangle$ 传播子做重整化完成的，也就是寻找带有两个外 $\mathcal{N}^i$ 腿的单圈图的发散，这和“(2+1) 维霍拉瓦引力中的渐近自由”一节中使用带外 $H_{ij}$ 腿的图不同。要重整化其余耦合常数，需要使用基于 Schwinger-DeWitt[1,46-50] 或 Gilkey-Seeley[74-76] 热核技术的更高效方法，及其以泛函迹通用方法 [48, 49] 形式给出的扩展。事实证明，为解决非极小和高阶导数算符的问题，上述扩展方法配合特殊维度约化变换是必要的。这些方法的应用过程如下。

For the renormalization of the potential part of the action (84), it is sufficient to consider the metric background on which all its five tensor structures are nonvanishing and can be distinctly separated. This is the spacetime metric with a generic static three-dimensional part $g_{ij}(\mathbf{x})$ and vanishing shift functions $N^i = 0$ . The static nature of $g_{ij}$ and zero shift functions lead to zero kinetic term of (84) whose contribution is not needed for the renormalization of couplings $\upsilon_1, \ldots \upsilon_5$ . The one-loop effective action on such a background is given by Gaussian integration over the full set of quantum fields $\left(h_{ij}, n^i, c^i, \bar{c}_i\right)$ . These fields in the $(\sigma, \xi)$ -family of background-covariant gauges (60)-(62) have the following quadratic parts of their gauge-fixed and ghost actions:

对于作用量 (84) 势能部分的重整化，只需考虑满足所有五个张量结构都非零且可明确区分的度规背景即可。该时空度量具有一般静态三维部分 $g_{ij}(\mathbf{x})$ ，且移位函数 $N^i = 0$ 为零。$g_{ij}$ 的静态特性以及零移位函数使得 (84) 的动能项为零，耦合 $\upsilon_1, \ldots \upsilon_5$ 的重整化不需要该动能项的贡献。该背景下的单圈有效作用量由全量量子场 $\left(h_{ij}, n^i, c^i, \bar{c}_i\right)$ 上的高斯积分给出。这些场在 $(\sigma, \xi)$ 族背景协变规范 (60)-(62) 中，其规范固定作用量和鬼作用量具有如下二次部分:

$$S_h = \frac{1}{2G} \int d\tau d^3x \sqrt{g} h_{mn} G^{mn,ij} \left[-\delta_{ij}{}^{kl} \partial_\tau^2 + \mathbf{D}_{ij}{}^{kl}(\nabla)\right] h_{kl}, \tag{86}$$

$$S_n = \frac{\sigma}{2G} \int d\tau d^3x \sqrt{g} n^i O_{ij} \left[-\delta_k^j \partial_\tau^2 + \mathbf{B}_k^j(\nabla)\right] n^k, \tag{87}$$

$$S_{\rm gh} = \frac{1}{G} \int d\tau d^3x \sqrt{g} \bar{c}_i \left[-\delta_j^i \partial_\tau^2 + \mathbf{B}^i{}_j(\nabla)\right] c^j. \tag{88}$$

It is important that for these gauges the $n^i h_{kl}$ cross term is absent in the quadratic part of the total action, and for a static background the spatial differential operator $\mathbf{B}^i{}_j(\nabla)$ is exactly the same in the shift and ghost parts of the action. The spatial differential operators here have the form,

重要的是，对于这些规范，总作用量的二次项中不存在 $n^i h_{kl}$ 交叉项，且对于静态背景，空间微分算子 $\mathbf{B}^i{}_j(\nabla)$ 在作用量的位移部分和鬼部分中完全相同。此处的空间微分算子形式如下,

$$\mathbf{D}_{ij}{}^{kl}(\nabla) = -\left[\upsilon_5 \delta_{ij}{}^{kl} + \frac{4\upsilon_4(1-\lambda)+\upsilon_5}{1-3\lambda} g_{ij} g^{kl}\right] \Delta^3 + \left[2\upsilon_5 - \frac{1}{\sigma}\right] \delta_{(i}^{(k} \nabla_{j)} \nabla^{l)} \Delta^2$$

$$+ \frac{4\upsilon_4(1-\lambda)+\upsilon_5}{1-3\lambda} g_{ij} \nabla^{(k} \nabla^{l)} \Delta^2 + \left[4\upsilon_4 + \upsilon_5 + \frac{\lambda(1+\xi)}{\sigma}\right] \nabla_{(i} \nabla_{j)} g^{kl} \Delta^2$$

$$-\left[4v_4+2v_5+\frac{\xi}{\sigma}\right]\nabla_{(i}\nabla_{j)}\nabla^{(k}\nabla^{l)}\Delta+\cdots, \tag{89}$$

$$\mathbf{B}^i{}_j(\nabla)=-\frac{1}{2\sigma}\delta^i_j\Delta^3-\frac{1}{2\sigma}\Delta^2\nabla_j\nabla^i-\frac{\xi}{2\sigma}\nabla^i\Delta\nabla^k\nabla_j\nabla_k$$

$$-\frac{\xi}{2\sigma}\nabla^i\Delta\nabla_j\Delta+\frac{\lambda}{\sigma}\Delta^2\nabla^i\nabla_j+\frac{\lambda\xi}{\sigma}\nabla^i\Delta^2\nabla_j,\ \Delta=\gamma^{ij}\nabla_i\nabla_j, \tag{90}$$

where in the metric sector we retain only the principal symbol part of the operator (its highest-order derivative terms), because the rest of it contains about 200 lower derivative terms. This directly points out to the level of calculational complexity of the problem, which even in the framework of background field and heat kernel approach can be solved only with the aid of xAct package of Mathematica program [44]. These terms are linear, quadratic, and cubic in spatial curvature and its covariant derivatives $\nabla_i$ (defined with respect to the background metric $g_{ij}$ - the notation used in contrast to $D_i$ above). The notation $G^{ij,kl}$ is used for the DeWitt metric (modified by generic $\lambda$ ) in the space of second-rank tensor fields

其中在度规部分我们仅保留算子的主象征部分 (即它的最高阶导数项)，因为算子的其余部分包含约 200 个低阶导数项。这直接说明了该问题的计算复杂程度: 即使在背景场方法和热核方法的框架下，也只能借助 Mathematica 程序的 xAct 包求解 [44]。这些项是空间曲率及其协变导数 $\nabla_i$ (相对于背景度规 $g_{ij}$ 定义，该记号是为了区别于上文的 $D_i$ ) 的一次项、二次项和三次项。记号 $G^{ij,kl}$ 用于表示二阶张量场空间中的 DeWitt 度规 (由一般的 $\lambda$ 修正)

$$G^{ij,kl}=\frac{1}{8}\left(g^{ik}g^{jl}+g^{il}g^{jk}\right)-\frac{\lambda}{4}g^{ij}g^{kl},\ \delta_{ij}{}^{kl}\equiv\delta^k_{(i}\delta^l_{j)}. \tag{91}$$

Gaussian integration over $\left(h_{ij},n^i,c^i,\bar{c}_i\right)$ gives the one-loop effective action

对 $\left(h_{ij},n^i,c^i,\bar{c}_i\right)$ 做高斯积分后得到单圈有效作用量

$$\Gamma_{\text{one-loop}}=\frac{1}{2}\operatorname{Tr}\ln\left[-\delta_{ij}{}^{kl}\partial_\tau^2+\mathbf{D}_{ij}{}^{kl}(\nabla)\right]-\operatorname{Tr}\ln\left[-\delta^i_j\partial_\tau^2+\mathbf{B}^i{}_j(\nabla)\right], \tag{92}$$

where the contribution of the gauge-fixing matrix $\mathcal{O}_{ij}$ is cancelled by the extra normalization factor (Det $\mathcal{O}_{ij})^{1/2}$ which comes from smearing the gauge-fixing conditions with a Gaussian weight (that generates the gauge-breaking term (30); see discussion in section ”BRST Structure of Renormalization and Covariance of Counterterms”), and the contribution of $G^{ij,kl}$ is cancelled by the local measure.

其中规范固定矩阵 $\mathcal{O}_{ij}$ 的贡献被额外归一化因子 Det $\mathcal{O}_{ij})^{1/2}$ 抵消，该因子来自用高斯权重弥散规范固定条件 (高斯权重生成了规范破缺项 (30)，参见“重整化的 BRST 结构和抵消项的协变性”一节中的讨论)，而 $G^{ij,kl}$ 的贡献被局域测度抵消。

Thus, the operators in (92), usually treated under the sign of functional trace logarithm by the heat kernel method, are both strongly nonminimal - their derivatives do not form a Laplacian or d’Alembertian - and go beyond the second order in derivatives. Therefore, they cannot be directly handled by the Schwinger-DeWitt heat kernel method which applies only to minimal second-order operators. The first step to circumvent this difficulty is to use the following dimensional reduction transform for the set of operators $\mathbf{F}=\left(\mathbf{D}_{ij}{}^{kl},\mathbf{B}^i{}_j\right)$, which is possible for static in time backgrounds,

因此，式 (92) 中的这些算子通常是通过热核方法处理泛函迹对数，它们不仅具有强非最小性——导数不构成拉普拉斯算子或达朗贝尔算子——而且导数阶数超过二阶。因此，它们无法直接通过仅适用于最小二阶算子的 Schwinger-DeWitt 热核方法处理。为解决这一困难，第一步是对算子集合 $\mathbf{F} = \left(\mathbf{D}_{ij}{}^{kl}, \mathbf{B}^i{}_j\right)$ 进行如下维数约化变换，这对不随时间变化的静态背景是可行的，

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}\ln\left[-\partial_\tau^2 + \mathbf{F}\right]\\
&= -\frac{1}{2}\int d\tau d^3x \int_0^\infty \frac{ds}{s}\,\mathrm{tr}\, e^{-s\left(-\partial_\tau^2 + \mathbf{F}\right)} \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')}\delta\left(\mathbf{x},\mathbf{x}'\right)\Big|_{\tau=\tau',\mathbf{x}=\mathbf{x}'}\\
&= -\frac{1}{4\sqrt{\pi}}\int d\tau d^3x \int_0^\infty \frac{ds}{s^{3/2}}\,\mathrm{tr}\, e^{-s\mathbf{F}}\delta\left(\mathbf{x},\mathbf{x}'\right)\big|_{\mathbf{x}=\mathbf{x}'}\\
&= -\frac{\Gamma(-1/2)}{4\sqrt{\pi}}\int d\tau d^3x\,\mathrm{tr}\sqrt{\mathbf{F}}\delta\left(\mathbf{x},\mathbf{x}'\right)\Big|_{\mathbf{x}=\mathbf{x}'} = \frac{1}{2}\int d\tau \mathrm{Tr}_3\sqrt{\mathbf{F}}.
\end{aligned} \tag{93}$$

Here tr denotes the matrix trace in the vector space of field indices and $\mathrm{Tr}_3$ is the functional trace on the space of functions of three-dimensional coordinates, as opposed to the four-dimensional $\mathrm{Tr} \equiv \mathrm{Tr}_4$ .

此处 tr 表示场指标向量空间中的矩阵迹，$\mathrm{Tr}_3$ 是三维坐标函数空间上的泛函迹，与之对应，四维坐标下是 $\mathrm{Tr} \equiv \mathrm{Tr}_4$ 。

To handle the operator square root we note that the sixth-order operators $\mathbf{F}$ have the form

为处理算子平方根，我们注意到六阶算子 $\mathbf{F}$ 具有如下形式

$$\mathbf{F}(\nabla) = \sum_{a=0}^{6} \mathcal{R}_{(a)} \sum_{6\geq 2k\geq a} \alpha_{a,k}\nabla_1 \ldots \nabla_{2k-a}(-\Delta)^{3-k},\ \mathcal{R}_{(a)} = O\left(\frac{1}{l^a}\right), \tag{94}$$

where $\mathcal{R}_{(a)}$ denote the local coefficients built of curvature tensor and its covariant derivatives of (physical) dimension $a$ in units of inverse length. Obviously, its square root is a pseudodifferential operator which, when expanded in powers of the curvature, becomes an infinite series in local curvature structures $\mathcal{R}_{(a)}$ of ever-growing dimensionality $a$ , such that the coefficient of every such structure is an operator polynomial in covariant derivatives of a finite order (determined by $a$ ) times a certain power of the covariant Laplacian $\Delta$ . Dimensional considerations suggest that such an expansion, containing a finite set of positive powers of $\Delta$ and an infinite sequence of its negative powers, looks as follows:

其中 $\mathcal{R}_{(a)}$ 表示由曲率张量及其协变导数构造的局域系数，其量纲 (物理量纲) 为逆长度单位下的 $a$ 。显然，它的平方根是一个伪微分算子，将其按曲率幂次展开后，会得到维数不断增长的局域曲率结构 $\mathcal{R}_{(a)}$ 的无穷级数，且每个这类结构的系数都是有限阶协变导数的算子多项式 (由 $a$ 确定) 乘以协变拉普拉斯算子 $\Delta$ 的某次幂。量纲分析表明，这种展开包含 $\Delta$ 的有限个正幂次和无穷多个负幂次，形式如下:

$$\sqrt{\mathbf{F}(\nabla)} = \sum_{a=0}^{\infty} \mathcal{R}_{(a)} \sum_{k\geq a/2}^{K_a} \tilde{\alpha}_{a,k}\nabla_1 \ldots \nabla_{2k-a}\frac{1}{(-\Delta)^{k-3/2}}, \tag{95}$$

where $K_a$ is some finite integer for any $a$ and $\alpha_{a,k}$ are some coefficients.

其中对于任意 $a$ ，$K_a$ 是某个有限整数，$\alpha_{a,k}$ 是对应的系数。

The sequence of these tensor structures $\mathcal{R}_{(a)}$ and coefficients $\alpha_{a,k}$ can be found by iterations. First, one writes the needed square root as a sum of the term of zeroth order in curvature $\mathbf{Q}^{(0)}$ and the perturbation $\mathbf{X}$, $\sqrt{\mathbf{F}} = \mathbf{Q}^{(0)} + \mathbf{X}$.$\mathbf{Q}^{(0)}$ is built solely in terms of covariant derivatives and the powers of $\Delta$ . Zeroth order $\mathbf{Q}^{(0)}$ can be chosen by the following procedure. Take the principal symbol $\mathcal{F}(p)$ of the operator $\mathbf{F}(\nabla)$ by discarding all its curvature terms and replacing all covariant derivatives $\nabla_i$ by $c$ -number (momentum) vectors $p_i$ ,

这些张量结构 $\mathcal{R}_{(a)}$ 和系数 $\alpha_{a,k}$ 的序列可通过迭代得到。首先，我们将所需的平方根写为曲率零阶项 $\mathbf{Q}^{(0)}$ 的和，微扰项 $\mathbf{X}$, $\sqrt{\mathbf{F}} = \mathbf{Q}^{(0)} + \mathbf{X}$.$\mathbf{Q}^{(0)}$ 完全由协变导数和 $\Delta$ 的幂次构造。零阶项 $\mathbf{Q}^{(0)}$ 可通过下述步骤选取: 舍去算子 $\mathbf{F}(\nabla)$ 的所有曲率项，将所有协变导数 $\nabla_i$ 替换为 $c$ 数 (动量) 矢量 $p_i$ ，得到主象征 $\mathcal{F}(p)$ ,

$$\mathcal{F}(p) = \mathbf{F}(\nabla)\big|_{\nabla \to p, \mathcal{R} \to 0}, \tag{96}$$

and extract the matrix square root out of $\mathcal{F}(p)$ . Then define $\mathbf{Q}^{(0)}$ by replacing back vector arguments $p_i$ by the covariant derivatives $\nabla_i$ with their arbitrary but fixed once and for all ordering,

并从 $\mathcal{F}(p)$ 中提取矩阵平方根。随后通过将矢量自变量 $p_i$ 替换回协变导数 $\nabla_i$ 来定义 $\mathbf{Q}^{(0)}$ ，协变导数的顺序任意但一经选定便固定不变,

$$\mathbf{Q}^{(0)}(\nabla) = \left[\mathcal{F}(p)\right]^{1/2}\Big|_{p \to \nabla}. \tag{97}$$

Then use the fact that the unknown perturbation operator $\mathbf{X}(\nabla)$ satisfies the equation

接下来利用未知微扰算子 $\mathbf{X}(\nabla)$ 满足下述方程这一性质

$$\mathbf{Q}^{(0)}\mathbf{X} + \mathbf{X}\mathbf{Q}^{(0)} = \mathbf{F} - \left(\mathbf{Q}^{(0)}\right)^2 - \mathbf{X}^2.$$

With this choice of $\mathbf{Q}^{(0)}$ the right-hand side here contains a ”source term” $\mathbf{F} - \left(\mathbf{Q}^{(0)}\right)^2 \sim [\nabla, \nabla] \sim \mathcal{R}$ which is at least linear in the curvature, because this difference can be nonzero only due to noncommutativity of covariant derivatives.

按照 $\mathbf{Q}^{(0)}$ 的这种选取方式，此处方程的右侧包含一个“源项”$\mathbf{F} - \left(\mathbf{Q}^{(0)}\right)^2 \sim [\nabla, \nabla] \sim \mathcal{R}$ ，该项至少是曲率的一阶项，因为只有协变导数不对易时，这个差值才会非零。

Discarding the $\mathbf{X}^2$ term in the right-hand side of this equation, we see that in the lowest order $\mathbf{X}$ satisfies the linear equation and has a solution linear in $\mathcal{R}$ . Including $\mathbf{X}^2$ and solving the equation by iterations then gives a systematic method of expanding the needed square root $\sqrt{\mathbf{F}}$ in powers of the curvature and its derivatives. This expansion will have the form (95) with all powers of $\Delta$ commuted to the right, which can always be done, because every commutator $[\nabla, \Delta] \propto \mathcal{R}$ generates extra power of the curvature.

舍去该方程右侧的 $\mathbf{X}^2$ 项后我们可以看到，在最低阶下 $\mathbf{X}$ 满足线性方程，且存在一个关于 $\mathcal{R}$ 的线性解。之后引入 $\mathbf{X}^2$ 并通过迭代求解方程，即可得到将所需平方根 $\sqrt{\mathbf{F}}$ 按曲率及其导数的幂次展开的系统方法。该展开会得到形式 (95)，且 $\Delta$ 的所有幂次都对换到右侧，这总是可以实现的，因为每一个对易子 $[\nabla, \Delta] \propto \mathcal{R}$ 都会额外生成一个曲率幂次。

Thus, in view of (93) the divergent part of the effective action (92) takes the form of the sum of (integrated over time) terms

因此，结合式 (93)，有效作用量 (92) 的发散部分可表示为 (对时间积分的) 下列项之和

$$\mathrm{Tr}_3\sqrt{\mathbf{F}}\Big|^{\mathrm{div}} = \sum_{a=2}^{6}\sum_{k}\widetilde{\alpha}_{a,k}\int d^3x\mathcal{R}_{(a)}(\mathbf{x})\,\nabla_1\ldots\nabla_{2k-a}\frac{1}{(-\Delta)^{k-3/2}}\delta(\mathbf{x},\mathbf{x}')\Big|^{\mathrm{div}}_{\mathbf{x}=\mathbf{x}'},$$

(98)

which are just the universal functional traces of [48]. The calculation of these UV divergences is based on the proper time representation and the heat kernel of the minimal second-order operator - the covariant Laplacian $\Delta$ ,

这些正是文献 [48] 中的泛函迹。这些 UV 发散的计算基于固有时间表示和最小二阶算子 (即协变拉普拉斯算子 $\Delta$ ) 的热核，

$$\nabla\ldots\nabla\frac{\hat{1}}{(-\Delta)^{\alpha}}\delta(\mathbf{x},\mathbf{x}')\big|^{\mathrm{div}}_{\mathbf{x}'=\mathbf{x}} = \frac{1}{\Gamma(\alpha)}\nabla\ldots\nabla\int_0^\infty ds s^{\alpha-1}e^{s\Delta}\hat{\delta}(\mathbf{x},\mathbf{x}')\big|^{\mathrm{div}}_{\mathbf{x}'=\mathbf{x}}.$$

(99)

The expansion at $s\to 0$ of the heat kernel $e^{s\Delta}\hat{\delta}(\mathbf{x},\mathbf{x}')$ in terms of the Synge world function $\sigma(\mathbf{x},\mathbf{x}')$ and HAMIDEW coefficients $a_n(\mathbf{x},\mathbf{x}')$ allows one to systematically isolate UV divergences of these quantities as the divergence of the proper time integral $\int_0^\infty ds/s$ , due to well-known coincidence limits of $\sigma(\mathbf{x},\mathbf{x}')$ , $a_n(\mathbf{x},\mathbf{x}')$ , and their multiple derivatives [1, 48] .

利用热核 $e^{s\Delta}\hat{\delta}(\mathbf{x},\mathbf{x}')$ 在 $s\to 0$ 处基于辛世界函数 $\sigma(\mathbf{x},\mathbf{x}')$ 和哈米德系数 $a_n(\mathbf{x},\mathbf{x}')$ 的展开，我们可以系统分离出这些量中的紫外发散——该发散来自固有时间积分 $\int_0^\infty ds/s$ 的发散，根源是 $\sigma(\mathbf{x},\mathbf{x}')$ , $a_n(\mathbf{x},\mathbf{x}')$ 众所周知的重合极限及其多阶导数 [1, 48] 。

With the identification

在对应关系

$$\int_0^\infty\frac{\mathrm{d}s}{s}\mapsto\log\left(\frac{\Lambda^2_{\mathrm{UV}}}{k_*^2}\right), \tag{100}$$

(it differs from (78) because the dimension of $s$ now is -2 ) this leads to the same Eqs. (79)-(80) for renormalized couplings and their beta-functions with very complicated set of functions $C_G$ and $C_{v_a}$ in $(3+1)$ -dimensional model. The resulting $\beta_\lambda$ (obtained in [43] by the diagrammatic method) reads

(它与式 (78) 不同，因为现在 $s$ 的维度是-2) 下，我们得到了和重整化耦合及其 β 函数完全相同的式 (79)-(80)，只是在 (3 + 1) 维模型中，函数集 $C_G$ 和 $C_{v_a}$ 形式极为复杂。通过图方法得到的最终 $\beta_\lambda$ (见文献 [43]) 为

$$\beta_\lambda = \frac{\mathcal{G}}{120\pi^2}\frac{27(1-\lambda)^2 + 3u_s(11-3\lambda)(1-\lambda) - 2u_s^2(1-3\lambda)^2}{(1-\lambda)(1+u_s)u_s}, \tag{101}$$

while the rest of beta-functions of essential couplings $\mathcal{G}$ and $\chi = (u_s, v_1, v_2, v_3)$ are given by the expressions

而基本耦合 $\mathcal{G}$ 和 $\chi = (u_s, v_1, v_2, v_3)$ 其余的 β 函数由以下表达式给出

$$\beta_{\mathcal{G}} = \frac{\mathcal{G}^2}{26880\pi^2}\frac{\sum_{n=0}^{7} u_s^n \mathcal{P}_n^{\mathcal{G}}[\lambda, v_1, v_2, v_3]}{(1-\lambda)^2(1-3\lambda)^2(1+u_s)^3 u_s^3}, \tag{102}$$

$$\beta_\chi = \frac{\mathcal{G}}{26880\pi^2}\frac{A_\chi \sum_{n=0}^{9} u_s^n \mathcal{P}_n^{\chi}[\lambda, v_1, v_2, v_3]}{(1-\lambda)^3(1-3\lambda)^3(1+u_s)^3 u_s^5}, \tag{103}$$

where $A_{u_s} = u_s(1-\lambda), A_{v_1} = 1, A_{v_2} = A_{v_3} = 2$ , and $\mathcal{P}_n^{\mathcal{G},\chi}[\lambda, v_1, v_2, v_3]$ is a set of very complicated polynomials of its arguments, which takes pages [44]. For illustration, an example of such a polynomial (one of the longest ones) is

其中 $A_{u_s} = u_s(1-\lambda), A_{v_1} = 1, A_{v_2} = A_{v_3} = 2$ ， $\mathcal{P}_n^{\mathcal{G},\chi}[\lambda, v_1, v_2, v_3]$ 是一组形式极为复杂的自变量多项式，其展开足足占用了数页篇幅 [44]。作为示例，我们给出其中一个最长的这类多项式如下

$$\begin{aligned}
&\mathcal{P}_5^{v_1}\\
&= -2(1-\lambda)^2(1-3\lambda)\{168v_2^3(51\lambda^3 - 149\lambda^2 + 125\lambda - 27) - 108v_3^3(9\lambda^3 + 9\lambda^2\\
&-25\lambda + 7) - 4v_2^2(1-\lambda)[18v_3(117\lambda^2 - 366\lambda + 109) - 284\lambda^2 - 7265\lambda + 5425]\\
&+40320v_1^2(1-\lambda)^2(\lambda+1) - 9v_3^2(3467\lambda^3 - 8839\lambda^2 + 6237\lambda - 865)\\
&+v_1[64v_2^2(1-\lambda)^2(1717\lambda - 581) - 16v_2(1-\lambda)(3v_3(2741\lambda^2 - 3690\lambda + 949)\\
&+25940\lambda^2 - 40662\lambda + 12022) + 27v_3^2(961\lambda^3 - 2395\lambda^2 + 1835\lambda - 401)\\
&+6v_3(52267\lambda^3 - 148963\lambda^2 + 129881\lambda - 33185) - 288353\lambda^3 + 542255\lambda^2\\
&-333355\lambda + 83485] - 2v_2[162v_3^2(3\lambda^3 + 35\lambda^2 - 51\lambda + 13) + 24v_3(1265\lambda^3
\end{aligned}$$

$$-2191\lambda^2+691\lambda+235)+30971\lambda^3-40323\lambda^2+13167\lambda-4451]$$

$$-12v_3\left(6551\lambda^3-11593\lambda^2+6124\lambda-1112\right)+109519\lambda^3-252396\lambda^2$$

$$+177357\lambda-34396\}\,. \tag{104}$$

## Dimensional Regularization in Hořava Models

## 霍拉瓦模型中的维数正规化

It is instructive to compare the Wilsonian type beta-functions (80), obtained by differentiation with respect to the running scale $k_*$, and beta-functions in minimal subtraction scheme of the dimensional regularization [77, 78], especially bearing in mind that the dimensions of coupling constants are defined now with respect to Lifshitz dimensionality rather than the physical one.

对比通过对跑动标度 $k_*$ 求导得到的威尔逊型 β 函数 (80)，与维数正规化最小减除方案中的 β 函数是有启发意义的，尤其要注意的是，现在耦合常数的维数是相对于李夫希茨维数而非物理维数定义的。

For generic multicharge theory with the dimensionless renormalized coupling constants $g=\left\{g^A\right\}, A=1,2,\ldots N$, their bare couplings are denoted as $g_0^A$ and have nontrivial dimensions which get regularized according to

对于含有无量纲重整化耦合常数 $g=\left\{g^A\right\}, A=1,2,\ldots N$ 的一般多荷理论，其裸耦合记为 $g_0^A$，具有非平凡维数，该维数按下式正规化

$$\left[g_0^A\right]=\Delta_A+\rho_A\varepsilon,\ \left[g^A\right]=0,\ \varepsilon=d_{\text{phys}}-d\to 0, \tag{105}$$

where $d_{\text{phys}}$ is a physical (integer) space dimensionality and $d$ is its regularization by analytical continuation into the complex plane. These dimensions are generically linear in $\varepsilon$ with specific coefficients $\rho_a$ [78]. The bare coupling is a series in powers of $1/\varepsilon$ as a function of renormalized couplings

其中 $d_{\text{phys}}$ 是物理 (整数) 空间维数，$d$ 是通过解析延拓到复平面得到的正规化维数。这些维数一般关于 $\varepsilon$ 线性，带有特定系数 $\rho_a$ [78]。裸耦合是 $1/\varepsilon$ 的幂级数，是重整化耦合的函数

$$g_0^A=\mu^{\Delta_A+\rho_A\varepsilon}\left(g^A+\sum_{n=1}^{\infty}\frac{a_n^A(g)}{\varepsilon^n}\right), \tag{106}$$

where $\mu$ is a running scale whose factor recovers a correct dimension of the bare constant. One loop approximation contributes a single pole in $\varepsilon$, while higher-order poles are due to multi-loop corrections.

其中 $\mu$ 是跑动标度，其因子恢复裸耦合常数的正确维数。单圈贡献仅含一个 $\varepsilon$ 极点，而高阶极点来自多圈修正。

A conventional assumption is that $g_0$ is independent of $\mu, (\mu d/d\mu)\, g_0^a = 0$ , and divergent at $\varepsilon \to 0$ , while the running renormalized charge $g = g(\mu, \varepsilon)$ is analytic at $\varepsilon \to 0$ . This leads to the definition of the beta-function which is also analytic in this limit, $(\mu d/d\mu)\, g^A(\mu, \varepsilon) = \beta^A(g(\mu, \varepsilon), \varepsilon)$ . Then, differentiating (106) and demanding the cancellation of the linear in $\varepsilon, \varepsilon^0$ , and $\varepsilon^{-1}$ terms, one has [77, 78]

常规假设 $g_0$ 与 $\mu, (\mu d/d\mu)\, g_0^a = 0$ 无关，在 $\varepsilon \to 0$ 处发散，而跑动重整化荷 $g = g(\mu, \varepsilon)$ 在 $\varepsilon \to 0$ 处解析。这就得到了在该极限下也解析的 β 函数定义 $(\mu d/d\mu)\, g^A(\mu, \varepsilon) = \beta^A(g(\mu, \varepsilon), \varepsilon)$ 。随后，对 (106) 求导并要求抵消线性的 $\varepsilon, \varepsilon^0$ 项和 $\varepsilon^{-1}$ 项，可得 [77, 78]

$$\beta^A(g, \varepsilon) = -(\Delta_A + \rho_A \varepsilon)\, g^A - \rho_A a_1^A(g) + \sum_{B=1}^{N} \rho_B g^B \frac{\partial a_1^A(g)}{\partial g^B}, \tag{107}$$

whereas the cancellation of higher-order poles results in recurrent equations for $a_n$ , $n > 1$ .

而高阶极点的抵消给出关于 $a_n$ 、$n > 1$ 的递推关系。

In the context of Hořava gravity $g^A \mapsto (G, v^a)$ , and $\lambda$ can be included into the set of $v_a$ just like it was done in Eq. (79) for $(2+1)$ -dimensional model. The dimension parameters of Eqs. (105)-(106) should be interpreted in terms of anisotropic Lifshitz dimensions. In particular, the Lifshitz dimension of coupling constants should be related to the regularized space dimensionality

在霍拉瓦引力的背景下，$g^A \mapsto (G, v^a)$ 和 $\lambda$ 可以被纳入 $v_a$ 集合，就像式 (79) 对 $(2+1)$ 维模型所做的那样。式 (105)-(106) 的维数参数应当按各向异性李夫希茨维数解释，尤其是耦合常数的李夫希茨维数应与正规化空间维数相关联

$$[G_0] = d_{\text{phys}} - d = \varepsilon,\ \left[v_a^0\right] = 0. \tag{108}$$

Note that the coupling constant $G$ differs from the rest of the couplings because the dimension of its bare version gets regularized. In accordance with Equation (105) this identifies its parameters to be $\rho_G = 1, \rho_a = 0$ , and $\Delta_G = \Delta_a = 0$ .

请注意，耦合常数 $G$ 与其余耦合不同，因为其裸量的维度会被正则化。根据式 (105)，可确定其参数为 $\rho_G = 1, \rho_a = 0$ 和 $\Delta_G = \Delta_a = 0$ 。

Within minimal subtraction scheme the pole parameter $\varepsilon$ is related to the UV cutoff of Eq. (79), $\ln\left(\Lambda_{\text{UV}}^2/k_*^2\right) = 2/\varepsilon$ , so that this equation can be rewritten in the form of Eq. (106)

在最小减除方案中，极点参数 $\varepsilon$ 与式 (79) 的紫外截断 $\ln\left(\Lambda_{\text{UV}}^2/k_*^2\right) = 2/\varepsilon$ 相关，因此该方程可以改写为式 (106) 的形式

$$G_0 = \mu^\varepsilon \left(G + \frac{4C_G G^2}{\varepsilon} + \dots\right),\ v_a^0 = v_a + \frac{4G\left(C_G v_a - C_{v_a}\right)}{\varepsilon} + \dots, \tag{109}$$

which means that $a_1^G = 4C_G G^2$ and $a_1^{v_a} = 4G\left(C_G v_a - C_{v_a}\right)$ . Therefore, according to (107) and in view of the fact that $a_1^G$ and $a_1^{v_a}$ are respectively quadratic and linear in $G$ (remember that one-loop $C_G$ and $C_{v_a}$ are $G$ -independent), one has

这意味着 $a_1^G = 4C_G G^2$ 和 $a_1^{v_a} = 4G\left(C_G v_a - C_{v_a}\right)$。因此，根据式 (107)，并考虑到 $a_1^G$ 和 $a_1^{v_a}$ 分别对 $G$ 是二次和一次关系 (注意单圈 $C_G$ 和 $C_{v_a}$ 与 $G$ 无关)，可得

$$\beta_G = \left[-\varepsilon G - a_1^G + G\frac{\partial a_1^G}{\partial G}\right]_{\varepsilon=0} = a_1^G = 4C_G G^2, \tag{110}$$

$$\beta_{v_a} = G\frac{\partial a_1^{v_a}}{\partial G} = a_1^{v_a} = -4GC_{v_a} + v_a\frac{\beta_G}{G}, \tag{111}$$

which fully agrees with the Wilsonian beta-functions (80). Both beta-functions again (as in Wilsonian approach) coincide with single pole residues of the bare constants in terms of renormalized ones, but this is achieved via nontrivial combination of terms in (107) and their special scaling in the perturbation theory constant $G$ .

这与威尔逊 beta 函数 (80) 完全一致。两种 beta 函数 (和威尔逊方法中一样) 都与裸常数按重整化常数表示的单极点留数重合，但这是通过式 (107) 中各项的非平凡组合及其在微扰论常数 $G$ 中的特殊标度实现的。

## Fixed Points

## 不动点

The number of coupling constants and complexity of their beta-functions thus far preclude from the full analysis of RG flows in $(3+1)$ -dimensional Hořava model. However, preliminary observations of their properties allow one to come to interesting conclusions and further prospects of this model.

耦合常数的数量及其 β 函数的复杂度目前阻碍了对 $(3+1)$ 维霍拉瓦模型中重整化群流的完整分析。不过，对其性质的初步观测已经能让我们得出有趣的结论，并展现出该模型的进一步研究前景。

An important question is the existence and nature of fixed points of the RG flow. The dependence of the $\beta$ -functions (101)-(103) on the coupling $\mathcal{G}$ factorizes, and this coupling determines the overall strength of interactions and must be small for the validity of the perturbative expansion. Its UV behavior determines whether the model is asymptotically free ($\mathcal{G} \to 0$) or has a Landau pole ($\mathcal{G} \to \infty$) . The rest of the couplings $\lambda, u_s, v_a$ are ratios of the coefficients in the action and need not be small. The search for fixed points of the RG flow thus consists in finding them for a subspace of the couplings $\lambda, u_s, v_a$ by solving the system,

一个核心问题是重整化群流不动点的存在性及其性质。式 (101)-(103) 的 $\beta$ 函数对耦合 $\mathcal{G}$ 的依赖是因式分解的，该耦合决定了相互作用的整体强度，且为了保证微扰展开的有效性必须取小值。它的紫外行为决定了该模型是渐近自由的 ($\mathcal{G} \to 0$) 还是存在朗道极点 ($\mathcal{G} \to \infty$)。其余耦合 $\lambda, u_s, v_a$ 是作用量中系数的比值，不必取小值。因此，搜索重整化群流的不动点就等价于在耦合 $\lambda, u_s, v_a$ 的子空间中通过求解下述方程组找到不动点，

$$\beta_\lambda/\mathcal{G} = 0,\ \beta_\chi/\mathcal{G} = 0,\ \chi = u_s, v_1, v_2, v_3, \tag{112}$$

and then evaluating $\beta_{\mathcal{G}}$ at a given solution, whose sign determines whether the flow trajectory goes to a Gaussian fixed point or a Landau pole. The results are summarized in Table 1. All these fixed points turn out to be asymptotically free, but the two last points correspond to very large values of $v_1$ and their validity should be taken with a certain reservation.

随后在给定解上计算 $\beta_{\mathcal{G}}$ ，其符号决定了流轨迹最终趋向高斯不动点还是朗道极点。结果总结在表 1 中。所有这些不动点均为渐近自由，但最后两个不动点对应 $v_1$ 的取值很大，其有效性需要保留一定保留。

It was conjectured in [79] that the UV fixed points of HG can be at infinite $\lambda$ and that the limit $\lambda \to \infty$ is interesting in cosmological applications. This turns out to be true - all $\beta$ -functions are finite at $\lambda \to \infty$ , whereas $\beta_\lambda$ is proportional to $\lambda, \beta_\lambda = -3\lambda\mathcal{G}\,(3-2u_s)\,/40\pi^2 u_s, \lambda \to \infty$ . Fixed points at $\lambda = \infty$ , which solve the equations $\beta_\chi/\mathcal{G}\big|_{\lambda=\infty} = 0$ for $\chi = u_s, v_1, v_2, v_3$ , are collected in Table 2. Three among them are UV attractive along the $\lambda$ -direction and correspond to asymptotically free fixed points. The structure of the RG flow around these points deserves a detailed study, which is also very interesting in the context of the Perelman-Ricci flows [80].

文献 [79] 曾推测，霍拉瓦引力的紫外不动点可存在于无穷大 $\lambda$ 处，且极限 $\lambda \to \infty$ 在宇宙学应用中具有重要意义。事实确实如此: 在 $\lambda \to \infty$ 处所有 $\beta$ 函数都是有限的，而 $\beta_\lambda$ 正比于 $\lambda, \beta_\lambda = -3\lambda\mathcal{G}\,(3-2u_s)\,/40\pi^2 u_s, \lambda \to \infty$ 。满足方程 $\beta_\chi/\mathcal{G}\big|_{\lambda=\infty} = 0$ 、对应 $\chi = u_s, v_1, v_2, v_3$ 的 $\lambda = \infty$ 处不动点汇总在表 2 中。其中三个在 $\lambda$ 方向上是紫外吸引的，对应渐近自由不动点。这些点附近的重整化群流结构值得深入研究，在佩雷尔曼-里奇流的背景下 [80] 这项研究也十分有意义。

Table 1 Solutions of the system (112). The sixth column gives the value of the $\beta$ -function for $\mathcal{G}$ at the respective solution and the seventh column indicates whether it corresponds to an asymptotically free fixed point. The eighth column tells if the fixed point is UV attractive along the $\lambda$ -direction

表 1 方程组 (112) 的解。第六列给出对应解处 $\mathcal{G}$ 的 $\beta$ 函数值，第七列标注该解是否对应渐近自由不动点，第八列标注该不动点在 $\lambda$ 方向上是否为紫外吸引

| $\lambda$ | $u_S$ | $v_1$ | $v_2$ | $v_3$ | $\beta_{\mathcal{G}}/\mathcal{G}^2$ | AF? | UV attractive along $\lambda$ ? |
|---|---|---|---|---|---|---|---|
| 0.1787 | 60.57 | -928.4 | -6.206 | -1.711 | -0.1416 | Yes | No |
| 0.2773 | 390.6 | -19.88 | -12.45 | 2.341 | -0.2180 | Yes | No |
| 0.3288 | 54533 | $3.798 \times 10^8$ | -48.66 | 4.736 | -0.8484 | Yes | No |
| 0.3289 | 57317 | $-4.125 \times 10^8$ | -49.17 | 4.734 | -0.8784 | Yes | No |

Table 2 Fixed points of Hořava gravity at $\lambda = \infty$

表 2 $\lambda = \infty$ 处霍拉瓦引力的不动点

| $u_s$ | $v_1$ | $v_2$ | $v_3$ | $\beta_{\mathcal{G}}/\mathcal{G}^2$ | AF? | UV attractive along $\lambda$ ? |
|---|---|---|---|---|---|---|
| 0.01950 | 0.4994 | -2.498 | 2.999 | -0.2004 | Yes | No |
| 0.04180 | -0.01237 | -0.4204 | 1.321 | -1.144 | Yes | No |
| 0.05530 | -0.2266 | 0.4136 | 0.7177 | -1.079 | Yes | No |
| 12.28 | -215.1 | -6.007 | -2.210 | -0.1267 | Yes | Yes |
| 21.60 | -17.22 | -11.43 | 1.855 | -0.1936 | Yes | Yes |
| 440.4 | -13566 | -2.467 | 2.967 | 0.05822 | No | Yes |
| 571.9 | -9.401 | 13.50 | -18.25 | -0.07454 | Yes | Yes |
| 950.6 | -61.35 | 11.86 | 3.064 | 0.4237 | No | Yes |

Another interesting observation is that for a special choice of the values

另一个有趣的结论是，对于特殊取值选择

$$\{v^*\} : v_1 = 1/2,\ v_2 = -5/2,\ v_3 = 3, \tag{113}$$

the limit $u_s \to 0$ of all $\beta$ -functions except $\beta_\lambda$ becomes regular for any $\lambda$ in the unitary domain, and, moreover, three beta-functions turn out to be zero, $\beta_{v_a}\big|_{\{v^*\},u_s\to 0} = 0, a = 1,2,3$ . The point $\{v^*\}, u_s \to 0$ is special since it corresponds to the version of HG , in which the potential term is a square of the Cotton tensor $C_{ij} = \varepsilon^{kl(i}\nabla_k R_l^{j)}, \varepsilon^{ikl} = \epsilon^{ikl}/\sqrt{g}, \epsilon^{123} = 1$ .

除 $\beta_\lambda$ 外，所有 $\beta$ 函数的极限 $u_s \to 0$ 在酉域内对任意 $\lambda$ 都变得正则，此外还有三个 β 函数恰好为零，即 $\beta_{v_a}\big|_{\{v^*\},u_s\to 0} = 0, a = 1,2,3$ 。点 $\{v^*\}, u_s \to 0$ 是特殊的，因为它对应势项为 Cotton 张量平方的 HG 形式 $C_{ij} = \varepsilon^{kl(i}\nabla_k R_l^{j)}, \varepsilon^{ikl} = \epsilon^{ikl}/\sqrt{g}, \epsilon^{123} = 1$ 。

$$S[g] = \frac{1}{2G}\int \mathrm{d}\tau \mathrm{d}^3x\sqrt{g}\left(K_{ij}K^{ij} - \lambda K^2 + v_5 C^{ij}C_{ij}\right). \tag{114}$$

This version of HG was originally suggested in [12] and its quantum properties were studied in [87]. It is known as HG with detailed balance and is interesting because the Cotton tensor can be rewritten as a variational derivative of the three-dimensional gravitational Chern-Simons action, $\sqrt{g}C^{ij} = -\delta W_{\mathrm{CS}}[g]/\delta g_{ij}(x)$ ,

这种 Hořava 引力形式最初由文献 [12] 提出，其量子性质由文献 [87] 开展研究。它以带详细平衡的 Hořava 引力为人所知，其有趣之处在于 Cotton 张量可以改写为三维引力陈-西蒙斯作用量的变分导数，$\sqrt{g}C^{ij} = -\delta W_{\mathrm{CS}}[g]/\delta g_{ij}(x)$

$$W_{\mathrm{CS}}[g] = \frac{1}{2}\int \mathrm{d}^3x\varepsilon^{ijk}\left(\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}\right), \tag{115}$$

defined in terms of the Christoffel symbol as a functional of $g_{ij}$ . Then the integrand of (114) can be rewritten as a square of the Langevin equation characteristic of stochastic quantization of three-dimensional gravity [81, 82]. Further, there exists a deformation of the action (115) by relevant operators which preserves the detailed balance structure and is related to the topological massive gravity [83-85]. The detailed balance relation between $d$ and $(d+1)$ -dimensional theories appears in the context of stochastic quantization and establishes a nontrivial connection between the renormalization properties of the two theories [86]. In our case this suggests an intriguing connection between the $(3+1)$ -dimensional projectable HG and the three-dimensional gravitational Chern-Simons/topological massive gravity [87].

该作用量以克里斯托费尔符号定义，是 $g_{ij}$ 的泛函。于是 (114) 的被积函数可以改写为朗之万方程的平方，这是三维引力随机量子化的特征 [81, 82]。此外，作用量 (115) 可由相关算符形变，该形变保留了详细平衡结构，且与拓扑有质量引力相关 [83-85]。$d$ 与 $(d+1)$ 维理论之间的详细平衡关系出现在随机量子化的框架中，它建立了两个理论重整化性质之间的非平凡关联 [86]。在我们的研究情形中，这表明 $(3+1)$ 维投影 Hořava 引力与三维引力陈-西蒙斯/拓扑有质量引力之间存在值得探究的关联 [87]。

It is important to emphasize, however, that the point $\{v^*\}, u_s \to 0$ is neither fixed nor fully regular point of the RG flow, because the $\beta$ -function of the remaining essential coupling $\lambda$ diverges in this limit, $\beta_\lambda|_{\{v^*\},u_s\to 0} \sim 1/u_s$ . Thus, the physical significance of the critical point (113) is unclear at the moment. It will be interesting to understand if the inclusion of fermionic degrees of freedom appearing in the stochastic quantization framework [87] can change the picture.

但需要着重指出，点 $\{v^*\}, u_s \to 0$ 既不是重整化群流的不动点，也不是完全正则点，因为剩余本质耦合 $\lambda$ 的 $\beta$ 函数在该极限下发散，即 $\beta_\lambda|_{\{v^*\},u_s\to 0} \sim 1/u_s$。因此，临界点 (113) 的物理意义目前尚不明确。探究纳入随机量子化框架中出现的费米自由度 [87] 是否会改变这一结论将是很有意义的。

## Conclusions and Discussion

## 结论与讨论

In this overview we have briefly exposed the current status of UV renormalization in Hořava gravity theory. This includes the renormalizability of its projectable models in all spacetime dimensions, subtle mechanism of a possible restoration of unitarity in their non-projectable version, demonstrates the origin of UV asymptotic freedom in $(2+1)$ -dimensional HG and points out to the existence of several fixed points of RG flow in $(3+1)$ -dimensional theory, which also might serve as good candidates for its UV asymptotic freedom. Along with the description of the calculational strategy for beta-functions of the latter theory, we dwelt on the generalized Schwinger-DeWitt technique of universal functional traces which strongly exceeds conventional Feynman diagrammatic method in its capacity to handle the models with broken Lorentz symmetry. The combination of this method with the heat kernel approach and background field formalism makes tractable computations in theories encumbered with a plethora of spacetime anisotropic structures inherent in HG.

在这篇综述中，我们简要阐述了霍拉瓦引力理论中紫外重整化的研究现状。内容包括任意时空维度下可投影模型的可重整性、非可投影版本中可能恢复幺正性的微妙机制，展示了 $(2+1)$ 维霍拉瓦引力中紫外渐近自由的起源，并指出 $(3+1)$ 维理论中存在多个重整化群流不动点，这些不动点也可能成为其紫外渐近自由的合理候选。在介绍后者 β 函数的计算方案之余，我们还讨论了通用泛函迹的广义施温格-德维特技术，该技术处理洛伦兹对称性破缺模型的能力远优于传统费曼图方法。将该方法与热核方法、背景场形式体系结合，就可以对霍拉瓦引力中大量固有时空各向异性结构的复杂理论进行计算。

In addition to implications of HG theory listed in Introduction, it is also worth mentioning other studies spanning several directions. As it has already been mentioned, in [87] the projectable version in $d = 3$ was considered with the detailed balance restriction on the parameters of the model [12]. This model is connected to three-dimensional topologically massive gravity via the stochastic quantization approach and it is argued that it inherits the renormalizability properties of the latter - the conclusion matching with the properties of beta-functions and their fixed points discussed in section ”One-Loop Beta-Functions of (3+1)-Dimensional Hořava Gravity.” However, the treatment of HG gauge invariance in [87] is somewhat obscure. The works [88-92] study the relation between HG and causal dynamical triangulations. In Ref. [93] a one-loop renormalization of a truncated version of the $d = 2$ projectable model was considered, but this type of truncation explicitly breaks gauge invariance of the theory. Finally, in Refs. [94] the one-loop counterterms for the gravitational effective action induced by a scalar field with Lifshitz scaling (see also [53, 95, 96] for earlier works on this subject) were computed. These counterterms were shown to have the structure of the bare HG action, which suggests that if pure HG is renormalizable, it remains so upon inclusion of matter. In fact this property follows from gauge-fixing procedure of section ”Renormalizability and the Problem of Irregular Propagators” - the choice of gauge for Lifshitz matter gauge symmetry-preserving regularity of propagators is also possible [40].

除引言中列出的霍拉瓦引力理论的相关推论外，还有多个方向的其他研究值得提及。正如前文所述，文献 [87] 在满足详细平衡对模型参数的限制下 [12] 研究了 $d = 3$ 维的可投影版本。该模型通过随机量子化方法与三维拓扑有质量引力相联系，并且论证了它继承了后者的可重整性——这一结论与“(3+1) 维霍拉瓦引力的单圈 β 函数”一节中讨论的 β 函数及其不动点性质一致。但文献 [87] 对霍拉瓦引力规范不变性的处理有些含糊。文献 [88-92] 研究了霍拉瓦引力与因果动力学三角剖分之间的关系。文献 [93] 讨论了 $d = 2$ 维可投影模型截断版本的单圈重整化，但这类截断明确破坏了理论的规范不变性。最后，文献 [94] 计算了具有李夫希茨标度的标量场诱导的引力有效作用量的单圈抵消项 (关于该课题的早期研究另见 [53, 95, 96] )。结果表明这些抵消项具有裸霍拉瓦引力作用量的结构，说明如果纯霍拉瓦引力是可重整的，加入物质后它仍然保持可重整。实际上这一性质可以从“可重整性与不规则传播子问题”一节的规范固定过程得到——对于李夫希茨物质，也可以选择保持规范对称性与传播子正则性的规范 [40]。

Besides applications in quantum gravity, Hořava gravity in $d = 2$ can govern the dynamics of membranes in M-theory [11]. Other applications include the holographic description of non-relativistic strongly coupled systems, analogous to those occurring in condensed matter physics [97, 98], the model of multilayer graphene [99].

除量子引力领域的应用外，$d = 2$ 维霍拉瓦引力还可以描述 M 理论中膜的动力学 [11]。其他应用包括对非相对论强耦合系统的全息描述，这类系统类似凝聚态物理中出现的系统 [97, 98]，以及多层石墨烯模型 [99]。

To finish this review of quantum Hořava gravity it is perhaps worth adding some comments relating its status to string theory. String theory provides a fruitful approach to the construction of a consistent theory of quantum gravity; see, e.g., [100]. But it makes this at the expense of introducing a very rich extra structure, complexity, and widely recognized nonuniqueness [101]. As opposed to this nonuniqueness and richness of string theory, overloaded with numerous extra structures, it makes sense to question if quantum gravity can be self-contained and consistent in a smaller framework. HG is an attempt to formulate such a framework constituted by the requirements of locality, renormalizability, and unitarity in (3+1) dimensions. Hořava gravity seems to be the only known example satisfying this set of requirements, and if it would be asymptotically free, that is, consistently complete in UV domain and extended to the realm of non-projectable models interpolating between low energy GR and Planckian physics, then it will have good chances of being the physical theory of our nature. The obtained results is a strong indication to the plausibility of this conjecture.

在结束这篇量子霍拉瓦引力的综述前，或许有必要补充一些关于它和弦理论关系的讨论。弦理论为构造自洽的量子引力理论提供了富有成效的思路，例如参见文献 [100]。但弦理论为此付出的代价是引入了极为丰富的额外结构、复杂度，以及公认的非唯一性 [101]。不同于弦理论这种充满非唯一性、内容丰富且承载大量额外结构的框架，我们有理由追问: 量子引力能否在更小的框架中保持自洽与完备？霍拉瓦引力就是在 (3+1) 维中，以局域性、可重整性和幺正性为要求构建这类框架的尝试。霍拉瓦引力似乎是目前已知唯一满足这一系列要求的例子，如果它确实是渐近自由的，也就是在紫外区域自洽完备，并且可以推广到连接低能广义相对论与普朗克物理的非可投影模型，那么它就很有希望成为描述我们自然的物理理论。目前已得到的结果强烈表明这一猜想是合理的。

Acknowledgments Original methods exhibited in this review would not be possible without strong thought-provoking influence of B. S. DeWitt and G. A. Vilkovisky to whom I am deeply grateful. I am also deeply grateful to D. Blas, M. Herrero-Valea, A. V. Kurov, S. M. Sibiryakov, and C. F. Steinwachs for long-term collaboration on the original results of this chapter, especially emphasizing Sergey Sibiryakov as a moving spirit behind the studies of Lorentz symmetry violating gravitational models. I would like to thank J. Bellorín, I. L. Buchbinder, J. Donoghue, M. Duff, G. Dvali, S. Fulling, A. Yu. Kamenshchik, E. Mottola, V. Mukhanov, S. Mukohyama, D. V. Nesterov, H. Osborn, N. Ohta, V. A. Rubakov, M. Sasaki, I. L. Shapiro, M. Shaposhnikov, S. Solodukhin, P. Stamp, A. A. Starobinsky, K. Stelle, A. A. Tseytlin, M. A. Vasiliev, W. Unruh, W. Wachowski, and R. Woodard for fruitful discussions.

致谢若不是受到 B. S. 戴维特与 G. A. 维尔科维斯基极具启发性的深刻影响，本文综述中展示的原创方法无法完成，在此我对二位致以深切谢意。同时，我也由衷感谢 D. 布拉斯、M. 埃雷罗-瓦莱亚、A. V. 库罗夫、S. M. 西比里亚科夫和 C. F. 施泰因瓦赫，感谢他们与我长期合作，共同取得本章的原创研究成果；尤其要指出，谢尔盖·西比里亚科夫是洛伦兹对称性破缺引力模型研究背后的核心推动者。我还要感谢 J. 贝洛林、I. L. 布赫宾德、J. 多诺霍、M. 达夫、G. 德瓦利、S. 富林、A. 尤·卡门施奇克、E. 莫托拉、V. 穆哈诺夫、S. 本山、D. V. 涅斯捷罗夫、H. 奥斯本、N. 太田、V. A. 鲁巴科夫、M. 佐崎、I. L. 沙皮罗、M. 沙波什尼科夫、S. 索洛杜欣、P. 斯坦普、A. A. 斯塔罗宾斯基、K. 施泰勒、A. A. 采特林、M. A. 瓦西里耶夫、W. 昂鲁、W. 瓦霍夫斯基和 R. 伍达德，感谢他们带来富有成效的讨论。

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